THE ATMOSPHERE

F. J. B. CORDEIRO

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ITS CHARACTERISTICS AND DYNAMICS

BY

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ERRATA

On page 46, line 43, should read $\dot{\psi} = C \sec^2 \vartheta$, instead of $\psi = C \sec^2 \vartheta$.

On page 47, line 15, should read

$$= 2\,\omega \sin \vartheta \sqrt{R^2\, \dot{\vartheta}^2 + R^2 \cos^2 \vartheta\, \dot{\psi}_r^{\,2}} = 2\,\omega \sin \vartheta\, v_r,$$

instead of

$$= 2\,\omega \sin \vartheta \sqrt{R^2\, \vartheta^2 + R^2 \cos^2 \vartheta\, \dot{\psi}_r^{\,2}} = 2\,\omega \sin \vartheta\, v_r.$$

On page 99, line 28, should read Fig. 24 instead of Fig. 25.

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To

Professor William Libbey

Of Princeton University

FOR EARLY ENCOURAGEMENT IN
THE STUDY OF THE CYCLONE

PREFACE

METEOROLOGY may be defined as the Physics of the Atmosphere. As such it is a purely mathematical science, although strangely enough its cultivators have usually not been mathematicians. This is undoubtedly the reason for its present backward condition in comparison with allied sciences.

Among the exceptions to the above rule Ferrel stands out prominently. This investigator, as far back as the fifties, attempted to place the science upon a dynamical basis. His work, admirable as it was, seems not to have been generally understood and often misapprehended by meteorologists. It was far from complete, as must necessarily be the case with all pioneer work. Ferrel, for instance, was not aware that a cyclone is dynamically a gyroscope.

It has for this reason seemed advisable to the author to attempt to correct and complete the work from the point where Ferrel left it, and it is hoped that this attempt has been in some measure successful. Since the work is necessarily purely mathematical, all methods and demonstrations have been presented in the simplest manner possible, so that they may be understood by the greatest possible number of readers.

The general circulation of the atmosphere, which is treated at some length, has been found to be essentially different from Ferrel's earlier as well as his later conceptions of it. The mechanics of certain non-meteorological phenomena, such as sound, and those coming under the head of light, electricity, etc., have been explained in a simple manner.

F. J. B. C.

BROOKLINE, MASS., November 8, 1909.

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THE ATMOSPHERE

CONSTITUTION

THE chief constituents of the atmosphere are nitrogen and oxygen, and these exist in the approximate proportions by volume of 78 for nitrogen to 21 for oxygen. These proportions are maintained with great regularity in all parts of the world and at all heights where it has been possible to obtain specimens of air for examination. This is as we should expect by reason of the diffusion of gases—a rather slow process—as well as from the convection due to the unceasing circulation of the atmosphere. The proportion is not absolutely constant, however, but has been found to vary for oxygen from 20.84 to 20.97 volumes in a hundred, the proportion seeming to decrease slightly with the height.

Carbonic dioxide is found everywhere in the lower regions of the atmosphere in the average proportion of 3 parts in 10,000 volumes. This proportion may be increased enormously in closed spaces inhabited by animals, as well as in volcanic regions, especially where there are limestone and coal-bearing strata. This gas is absorbed by plants in building up their structures, and in doing so they fix the carbon and liberate the oxygen again, usually volume for volume of the carbonic dioxide. Animals, on the contrary, inhale oxygen and exhale carbonic dioxide from their tissues, but there is of necessity no balance between these two processes in plants and animals, as has sometimes been supposed. The total amount of CO_2 in the atmosphere at any one period is too vast to be sensibly affected by such interchanges. On the other hand, it is not to be supposed that the composition of the atmosphere has not changed with the ages, and there seems to be good reason for believing that in times past the proportion of CO_2 was much greater than at present.

Ammonia in minute traces is found in the atmosphere and is chiefly of organic origin. It is best detected in rain water, and after a shower the air is practically clear of it. This ammonia, derived from organic substances, partly returns to them to supply their nitrogen. The amount in the atmosphere varies from .1 to 100 volumes per million.

Traces of nitrous and nitric acids are found, which probably are always manufactured whenever there is a discharge of lightning. Cavendish was able to produce these substances from atmospheric air by the passage of an electric spark. In the atmosphere these acids are often found in combination with the ammonia. Besides nitrous and nitric acids, ozone is produced in the track of lightning, but being very unstable its existence is only tempo-

rary, so that, according to Lord Rayleigh, it is not to be considered a constituent of the atmosphere. On the other hand, minute quantities of H_2O_2, or hydrogen peroxide, are constantly present.

Sir William Ramsay has analyzed atmospheric air in the following manner. The air is first passed through a tube full of a mixture of caustic soda and lime to remove the CO_2, and then through a U tube containing strong sulphuric acid to deprive it of its water and ammonia. Afterwards it is led over filings of red hot copper, which unite with the oxygen, and the nitrogen and the members of the Helium group, being inert, pass on.

To separate the nitrogen, the gases are passed over red hot magnesium, or better, a mixture of magnesium powder and lime, which gives calcium. The magnesium, or the calcium, unites with the nitrogen, and the inert gases pass on.

To separate these last gases from each other, they are first compressed in a bulb and cooled to —185° C. by being immersed in liquid air. The Argon, Krypton and Xenon condense to a liquid with the Helium and Neon dissolved in it. On removing the bulb from the liquid air, its temperature rises, and the Helium and Neon escape first, mixed with a large amount of Argon. Argon distils next, and the Krypton and Xenon remain to the last. By repeating this process of fractional distillation, the Argon, Krypton and Xenon can be separated from each other and from the Helium and Neon, the two latter remaining mixed together all the time. To separate Neon from Helium recourse must be had to liquid hydrogen. By this means the mixture of Helium and Neon can be cooled to —252° C., when the Neon becomes solid while the Helium still remains gaseous. Helium, the most refractory of all gases, has been liquefied. Its boiling point is about —267° C. or 6° absolute.

One hundred volumes of air contain only .937 volume of these inert gases, and by far the larger portion of this is Argon, which exists in the proportion of not quite 1% by volume in the atmosphere. The amount of the other four gases taken together is about $\frac{1}{400}$ of that of Argon.

The spectrum of the Aurora contains lines of Argon, Krypton and Xenon, the Krypton lines being most clearly and uniformly seen. The yellow-green line of the aurora, which has long been known as characteristic of it and only recently identified, is with little doubt identical with the yellow-green line of Krypton, their wave lengths being respectively .5571 and .5570μ.

Besides these regular constituents of the atmosphere, certain extraneous solid substances are so constantly present as to necessitate their being considered a part of the atmosphere. Although any solid substance in the air must, on the whole, be constantly falling, still they are maintained aloft by currents at times for indefinite periods. The dust which is swept from the surface of the earth by winds probably never stays suspended for any length of time. That which is shot up by volcanoes to very great heights, especially the im-

palpable dust resulting from the condensation of volcanic gases at great heights and low temperatures (SO_2), may remain suspended for much longer periods. But the constant rain of meteoric bodies, which though mostly of microscopic dimensions, are moving at the rate of about twenty miles a second when they meet the atmosphere and are consequently sublimated at the highest levels, keeps the atmosphere as a whole supplied with a constant proportion of falling solid constituents. Although this cosmic dust, the particles of which may vary from 1 mm. to .001 mm. in diameter, is relatively small, still in the aggregate it is considerable. It is sufficient to color freshly fallen snow on mountain tops and in the polar regions. It is composed principally of iron with traces of nickel and cobalt—all paramagnetic substances. Such dust is easily swept up with a magnet, and in fact the greater amount of it in the polar regions than elsewhere is due to the magnetic attraction of the earth's poles. It is probably effective in intensifying auroral and other magnetic phenomena and also, in conjunction with oxygen, which is also a paramagnetic substance, in increasing the permeability of the atmosphere.

Lastly we have to consider the presence of aqueous vapor. This gas is always present near the surface of the earth, though in exceedingly variable proportions, while at very moderate levels it is almost entirely lacking. We shall have much to say of this very important constituent. In fact, from all points of view, the three gases of practical importance in the atmosphere are oxygen, nitrogen and aqueous vapor.

The constituents of the atmosphere are, therefore,

	Vols. in 100	Atomic Weight	
Nitrogen..	78	14	Liquefies at —194° C. Solidifies —214° C.
Oxygen...	21	15.882	Liquefies at —180.5° C.
Argon....	1	39.9	Liquefies at —186.9° C. Solidifies —189.6° C.
CO_2......	.03		Liquefies at —95.5° C. Solidifies —107.5° C.
Krypton..	Trace	81.5	
Xenon....	"	128	
Neon.....	"	20	
Helium...	"	4	Liquefies at —267° C.
Hydrogen.	"	1.007	Liquefies at —252° C. Solidifies —258° C.
SO_2......	"	Near volcanoes.	
H_2O_2.....	"		
H_2NO_2...	"		
H_2NO_3...	"		
NH_3.....	"		
H_2O.....	Variable.	From a trace to 3.5 volumes.	

Besides which the following solid substances are usually found floating in the atmosphere, all in minute traces.

Iron, Nickel, Cobalt	Cosmics.	Calcium Carbonate, Magnesium Carbonate, Silica, Aluminum, Na_2SO_4, Organic matter and living germs	From the surface of the earth.
NaCl.	From the sea.		

TEMPERATURE AND DENSITY

It is easy to calculate the total mass of the atmosphere, as was done long ago by Pascal, but such huge figures have little practical significance. If we say that in round numbers the total amount of air attached to the earth is 5,500 million million tons, we derive little benefit from this information. Of more importance is the determination of its height, its shape and its characteristics at different levels. To do this we must be guided by the following laws, which have been derived from experiment. The principal law which applies to all perfect gases is expressed by the formula $p\,v = R\,\vartheta$ where p is the pressure, v the volume, ϑ the temperature, and R a constant depending upon the units selected. This is called the characteristic equation of a perfect gas and, although when far removed from their condensation points, gases follow this formula rather closely, still no gas does so exactly. In other words, no gas is a perfect gas. By a convenient, though strictly improper use of the term, a gas is spoken of as being more or less perfect according as it follows this formula more or less closely.

The air, like all material substances, is subject to the attraction of the earth; consequently the particles at any level are not only subjected to this force, but are also pressed by the superincumbent weight of all the higher layers. The pressure is, therefore, a maximum at the surface of the earth and zero at the superior limit. To determine the density of the air at any particular point, it is not only necessary to know the pressure, but also the temperature at that point, and since the temperature of the atmosphere is due practically entirely to the sun, the amount of heat received from the earth being infinitesimal by comparison, we shall begin our study of temperatures with the sun.

With regard to radiant heat, it has been found that the amount of such heat absorbed or emitted by a body depends greatly upon the nature of the surface bounding it. If such a surface reflects a large part of the heat falling upon it, whether from the outside or inside, it will absorb or emit a correspondingly lesser proportion of this heat. A surface thickly coated with lamp black reflects, regularly and irregularly, an insignificant amount of the radiant energy falling upon it and is taken as a practical realization of a fully absorbing or fully radiating substance. Such a substance has been called a "black body," but since under the influence of intense heat it may become white hot, it seems better to refer to it as a "fully radiating substance."

For a long time it was sought to find a law connecting the outflow of energy from a fully radiating body with its temperature. Stefan was the first to suggest a formula which agreed at all satisfactorily with experimental results, and we shall call this formula Stefan's Law. It is this—that the stream of energy, or amount of energy radiated in a second from a fully radiating body, is proportional to the fourth power of its temperature reckoned from the absolute zero. Experimental determinations of the stream of energy issuing from a fully radiating body were first carried out by Pouillet and Professor Herschel.

In determining the amount of energy received from the sun at the surface of the earth, there are great difficulties, chief of which is the calculation of the amount which is absorbed and reflected by the layers which the sun's rays have traversed. We may, however, assume that we are not far from the truth when we take the amount of energy from the sun falling vertically upon one square centimeter of a fully absorbing surface just outside the earth's atmosphere, as $\frac{1}{24}$ of a calorie per second. That is to say, if we had a cube each face of which was one square centimeter and one of these faces was blackened and exposed perpendicularly to the sun's rays, and if the cube were filled with water, its temperature would rise at first $\frac{1}{24}$ of a degree C. every second. Now the area of a sphere around the sun at the distance of the earth is 46,000 times the area of the sun's surface. The stream of energy from one square centimeter of the sun's surface is, therefore, 46,000 $\times$ $\frac{1}{24}$ calories, or 1920 calories per second.

Kurlbaum, who has made determinations of the amount of energy issuing from a fully radiating surface, has prepared the following table:

RATE OF FLOW OF ENERGY FROM 1 CM.² OF FULLY RADIATING SURFACE.

Absolute Temperature	Grammes of Water Heated 1° per second, or Calories
0°	0.000000
100°	0.000127
300°	0.010300
1000°	1.270000
3000°	103.000000
6000°	1650.000000
6250°	1930.000000

We see from this table that the amount of energy streaming from the sun's surface corresponds approximately to a temperature of 6250° absolute or about 6000° C. Now, if we suppose a sphere of 1 cm.² cross-section furnished with a fully absorbing surface at the distance of the earth, such a sphere will have 4 cm.² of surface. It is receiving from the sun $\frac{1}{24}$ calorie every second, and when it has arrived at a state of equilibrium it will radiate every second exactly this amount. Consequently the radiation per cm.²

will be $\frac{1}{96}$ of a calorie, or .0104 calorie per second. From Kurlbaum's table we see that this corresponds to a temperature of 300° absolute, or 27° C. The average temperature of the surface of the earth, as usually estimated, is 16° C., so that if we suppose that about 11° has been lost through absorption and reflection from the upper layers of the atmosphere, we find here a tolerable confirmation of our calculation. Langley, from his bolometric measurements, has estimated that 41% of the sun's energy fails to reach the surface of the earth directly. It has also been found that the radiation from the sun is not constant (Langley and Abbott), but is subject to fluctuations—amounting at times to possibly 20% of the total.

The temperature at the earth's surface has been measured at so many times and places that we can give very accurate averages for many places, and in fact have a very good idea of the distribution of temperatures over the whole globe. These temperatures vary greatly, of course, with place and season, but at any one place the average for a year remains remarkably constant.

The highest known temperature of the air (in shade) was taken at Murzouk, in Africa, and was 56.4° C. But a thermometer buried in the soil in Africa has given 74° C. Among the greatest degrees of cold ever registered is —60° C. at Semipalatinsk in Siberia, and —56.5° C. at Fort Reliance, in North America. Captain Nares recorded a temperature of —65° C. The following table, compiled from Buchan's isothermal charts, will give a good idea of the average temperature at various latitudes.

TEMPERATURE CENTIGRADE.

Latitude	January	July	Mean of Year
+80	—31.9	+1.0	—15.5
70	—26.5	6.9	— 9.8
60	—16.9	13.8	— 1.6
50	— 6.0	18.6	+ 6.3
40	+ 4.5	22.8	13.6
30	12.9	26.6	19.8
20	21.7	29.0	25.3
10	25.9	28.4	27.2
0	27.3	26.1	26.7
—10	29.9	24.0	25.9
20	26.6	20.8	23.7
30	23.0	15.6	19.3
40	17.6	11.1	14.4
50	11.1	6.4	8.8
60	3.6	0.0	1.8

According to more recent determinations by Hann, the average temperatures in high southern latitudes are somewhat less than those given above. As we have said before, the average temperature of the whole surface of the earth is about 16° C.

While our knowledge of temperatures at the surface of the earth is so definite, we have as yet only general ideas as to what it is at higher levels. It is easily noted that the temperature on the whole falls as we ascend—perhaps something like .4° C. for each hundred meters in the lower layers. But there is no regular law, and the rate of fall becomes less as we ascend. There is at times an "inversion" of temperature, noticeable especially at night, when owing to the rapid radiation from the earth's surface, the layers of air in close proximity to the earth are much colder than those above. It may happen in balloon ascensions that the passage is made through alternate layers of hotter and colder strata.

The thermometry of the upper atmosphere is attended with considerable difficulties owing to the disturbance produced by radiation, either directly from the sun or from surrounding objects. Thus, although the air itself has a very definite temperature, which falls considerably as we attain to great heights, yet a fully absorbing body exposed to the rays of the sun would increase its temperature until at the upper limit it would acquire a temperature of 27° C., while the surrounding air, what was left of it, would be somewhere near the absolute zero. A climber on a lofty mountain may suffer from the heat when exposed to the rays of the sun, although he is walking on ice and the actual temperature of the air is below freezing. For this reason the earlier data derived from balloon ascensions were valueless because, even though screened from the direct rays of the sun, they were not protected from the radiation of the balloon itself, which sometimes became much hotter than the surrounding air. Probably the best way of taking such readings would be to enclose the thermometer in a vessel with double sides, having a vacuum between them, and the sides silvered, like the receptacles in which liquid air is carried.

Great heights have been reached by manned balloons. Dr. Berson, of Berlin, ascended to 30,000 ft. in 1894. The barometer was nine inches and the temperature —48° C.

Dr. Berson and Professor Süring, in 1901, attained to a height of 10,300 meters (nearly 6½ miles). The recorded temperature was —40° C. This is the greatest authentic height ever reached by man. Glaisher and Coxwell, in 1862, became unconscious at 29,000 ft., after which the balloon was supposed to have risen to 36,000 ft., but this is uncertain. Berson and Süring remained conscious by inhaling oxygen.

Within recent years considerable progress has been made in obtaining temperatures at great heights by means of kites (Rotch) and sounding balloons (Ballon-Sondes)—small balloons to which are attached automatically

registering instruments. A German balloon of this type, called the Cirrus, on one occasion rose to a height of 54,000 ft., and registered a minimum temperature of —52.5° C. On another occasion it rose to a height of 61,000 ft. with a minimum temperature of —61° C., and on still another occasion it rose to 72,000 ft. (13¼ miles) and brought down a temperature of —35° C.

A balloon released at Tromsö, Norway, reached a height of ten miles, registering a temperature of —60° C. at a little over six miles, and a temperature of —47° C. at ten miles.

Finally, a balloon released at Uccle in Belgium, in the winter of 1908, rose to a height of 95,250 ft., or eighteen miles, recording a minimum pressure of two-fifths inches of mercury and a temperature of —67° C. at eight miles, and a temperature of —64° C. at eighteen miles.

From the fact that balloons have frequently registered a greater temperature at very great heights than at intermediate points, it has been supposed that there is a permanent inversive layer at those heights where the temperature after steadily falling begins to rise again (Hergesell, Arrhenius). But the fact that no *à priori* reason can be adduced for such a phenomenon renders it extremely doubtful, if not improbable. The effect of radiation vitiates somewhat all these measurements. A balloon exposed to the rays of the sun after it has risen through the greater part of the atmosphere and completely through the blanket of aqueous vapor, which possesses the strongest absorptive power, is practically unscreened. The balloon itself, no matter what its height, becomes quite hot under these conditions, and will exert an influence on its thermometers even though they are screened. While it is possible that such a permanent inversive layer exists at great heights, considerable more proof will have to be adduced than is at present at hand for its acceptance.

From the soundings which have already been taken we can get something of an idea of the distribution of the average isothermal surfaces in the upper air, but much remains to be done before our knowledge can in any sense be considered accurate.

Fig. 1 gives a rough indication of this distribution. The temperatures are Centigrade and the heights above the earth's surface are given in miles.

HEIGHT

The height to which the atmosphere extends has been the subject of much speculation and various estimates. That there is an upper limit is evident from various considerations. It is not necessary to suppose, as has been done, that after reaching a certain degree of rarefaction a gas loses all power of expansion and becomes as it were an extremely tenuous liquid. With our best air pumps, which reduce the pressure to a small

fraction of a millimeter, it is always possible to exhaust a little more, and there is no evidence of a limit being approached. There is, however, a universal ether which pervades all space and which, being a material substance, is supposed to have a definite density. We know very little about this ether, and some very vague and rather contradictory views have been entertained regarding it. As it carries all radiations, serving as a medium of communication between distant bodies, it seems certain that it is a material substance

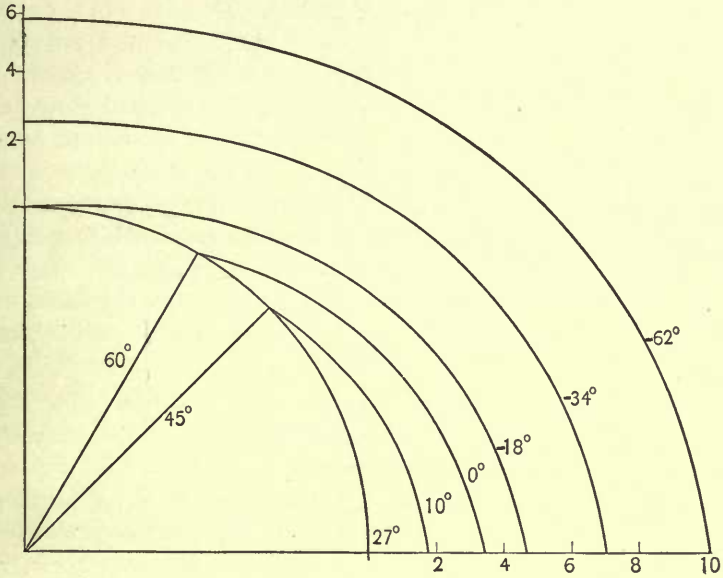

FIG. 1

endowed with the properties of elasticity and inertia of ordinary matter. It must also be the medium through which attraction (or repulsion) is exerted between two bodies, unless we suppose an independent medium for this purpose, which is not at all logical. Graetz has estimated the density of the ether as 9×10^{-16} that of water. Lord Kelvin, while giving no definite value, has calculated that it must be greater than 10^{-18}.

Another possible estimate may be arrived at as follows. The velocity of a disturbance in this medium, which is the velocity of all radiations, viz., 300,000 kilometers per second, should be $\sqrt{\frac{E}{D}}$, where E is its elasticity and D its density. Now the elasticity or pressure of this medium may possibly be inferred from magnetic attraction. If, as seems probable, magnetic lines of force are whirls in the ether which by centrifugal force produce a partial ether vacuum along their axes, just as a vortex in a fluid produces a partial vacuum, then the attraction between two magnets is to be explained as the

general pressure of the ether tending to destroy this vacuum. The greatest magnetic force which it has been possible to produce is about 400 lbs. to the square inch, which we may suppose to be an approach to a perfect ether vacuum, and consequently to indicate the general pressure of the ether.

Now 400 lbs to the square inch is equivalent to 27,577,607 dynes per square centimeter. Consequently we may write *$3 \times 10^{10} = \sqrt{\frac{27,577,607}{D}}$ where our values, being expressed in centimeters and grammes, will give the density in grammes per cubic centimeter.

This gives us somewhat less than $\frac{1}{3} \times 10^{-14}$ grammes per c.c., or in round numbers the density of the ether is $\frac{1}{3} \times 10^{-14}$ that of water.

It must be remembered that all this is highly hypothetical and scarcely susceptible of proof. But whether we take Graetz's estimate of roughly 10^{-17} times that of water, or our present estimate of $\frac{1}{3} \times 10^{-14}$, it is certain that the air must attain this extreme rarity at a comparatively short distance from the earth. It is difficult to conceive any atmosphere less dense than the ether as belonging to the earth, and apart from this there is some reason for thinking that if gravitation is effected through the ether, then any matter less dense than this medium would be repelled instead of attracted by a heavy body.†

Gravitation must be effected through some medium filling the space between two distant bodies, for if this space were an absolute vacuum no interaction of any kind could take place between them. As we have said before, having already one medium, viz., the ether, it would be illogical to postulate still another special medium by which gravitation is effected. We have been forced to consider the ether a material substance having at least two of the properties of ordinary matter. It must possess elasticity and inertia in order to transmit waves, though we cannot conceive it to have weight or temperature. We can conceive it to be full of waves of radiational (gravitational) energy, but it is the bearer of this energy merely. We cannot well suppose that it has any temperature itself or that it exerts of itself any attractional or repulsional force on other bodies or on itself or is attracted or repelled by them. Now it can be shown, both theoretically and experimentally, that when a heavy body sends waves through an elastic medium, all bodies denser than the medium which are swept by these waves will be attracted, while less dense bodies will be carried along *with* the waves, the limiting velocity in this case being the velocity of wave propagation. (v. Article Gravitation. l. c.)

According to this view, when the air has reached a limiting density of $\frac{1}{3} \times 10^{-14}$, or the density of the ether, it would no longer be attracted by

* 3×10^{10} centimeters per second is the velocity of light.

† See Article Gravitation, *Popular Astronomy*, January, 1905.

the earth but would be driven off with a high velocity. Owing to the extreme tenuity of the air at this limit, the rate of loss is very small and it can be easily calculated that it would take many millions of years for a complete dissipation of the atmosphere. That the earth is continually losing minute quantities of its atmosphere seems quite probable judging from the analogy of other planets. Thus the amount of the atmosphere surrounding the earth is considerably greater than that of Mars or the Moon, the latter seeming to have lost nearly all, while on the other hand it is less than that of Venus or the outermost planets. (v. Article The Atmosphere, *Popular Astronomy,* August, 1905.) When a wave sweeps over an aggregation of molecules, portions of the wave will be mutually reflected among them, some of the molecules being shielded by others from the wave. For a sufficient depth of the wave, the projections of the molecules upon a section perpendicular to the wave would form a practically continuous surface. To be added to this are the mutual interactions between the molecules or their resistance to being crowded together. For all these reasons, therefore, given a certain distribution of molecules to volume, the action of the wave on them will be the same as on a simple body distributed continuously over the same space with a uniform density. This uniform density will be the quotient of the sum of the masses of all the molecules divided by the space.

There is considerable difficulty in accepting the modern theory of radiation pressure. That light exerts pressure is experimentally demonstrated by causing it to strike upon a light vane in a vacuum, in which case the vane is repelled (Lebedew, Nichols). The experiment proves beyond a doubt that the ether has both mass (inertia) and elasticity and hence is a material substance, but beyond this proves very little. That the ethereal waves falling upon a surface much larger than the waves and being reflected from it should exert pressure, was to be expected if the ether is a material substance.

The theory supposes that the pressure is simply proportional to the surface exposed, while the gravitational force which is taken as always attractive is proportional to the mass. As for the same substance the mass is as the cube of its linear dimensions, while the surface is as the square, it follows that the repellent pressure is to the gravitational attraction inversely as the linear dimensions. Hence, by reducing the size of the body a limit will be reached where the repulsion is counterbalanced exactly by the attraction. If the body becomes still smaller the light pressure will overcome the force of attraction and the body will be driven off. It is calculated that a sphere having a diameter of $\frac{1}{100000}$ of an inch would be driven away from the sun by its light, which in this case is more powerful than the gravitational attraction. Whether in case the sphere were a "black body," and, therefore, did not reflect any light, the results would be changed, has not been considered. The theory is objectionable in that it postulates a double set of forces, one a repulsional force based upon an experiment in reflection, while

the force of gravitation remains as mysterious and unexplained as before and necessarily having no connection with ethereal vibrations.

It is further untenable from the fact that the molecules in the atmospheres of both the sun and the earth are far below the limit where the radiation pressure exceeds the gravitational attraction. Hence, were the theory true, the atmospheres of all bodies would immediately fly off, leaving them bare.

The attempt, therefore, to explain repulsional phenomena (comets' tails) as a differential effect between radiation pressure and gravitational attraction fails. Rather does it seem more consistent with facts and more supportable by theory, instead of multiplying actions, to find in the single action of gravitation alone the force which either attracts or repels according as the body acted upon is denser or rarer than the medium which carries the action.

In this connection we may briefly mention the theory of Dr. G. Johnstone Stoney, by which he attempts to account for the absence of hydrogen in the atmosphere. It is supposed, though there is no direct reason for thinking so, that the atmosphere formerly must have contained much more hydrogen than at present. It is now a mere trace. According to the kinetic theory, a hydrogen molecule is supposed to have an average velocity of $1\frac{1}{4}$ miles per second at 0° C. A molecule of oxygen at the same temperature is supposed to have an average velocity of rather less than one-third mile per second, and so on for the other gases, the velocity decreasing as the density increases. But these are merely the average velocities: some molecules are supposed to move faster, others slower, and a considerable proportion of hydrogen molecules may be conceived as having a higher velocity than seven miles per second. Now, this velocity, directed away from the earth, is sufficient to carry a body beyond the attraction of the earth into space, provided it is at the upper limit of the atmosphere. During the lapse of ages, Dr. Stoney supposes that in this way a considerable mass of hydrogen, practically all that the earth once possessed, has been lost from our atmosphere, as well as a lesser proportion of oxygen and nitrogen, etc. But it seems to have been forgotten that at the upper limit of the atmosphere the temperature falls nearly to that of space, which is probably not far from the absolute zero. Thus at a very short distance from the earth's surface the average velocity of a hydrogen molecule would be, not $1\frac{1}{4}$ miles a second, but a minute fraction of this amount, and the proportion of molecules having a velocity of seven miles would be infinitesimally small. The most probable supposition is that at the upper surface of the atmosphere the temperature is actually the absolute zero, so that it would be impossible for any gas to escape actively from the earth. As we have already pointed out, it is very probable that there is a dissipation of the atmosphere, but the process is a passive one, the gases being driven off by the action of the earth.

We have assumed a natural limit for the atmosphere, beyond which it

cannot extend, viz., the point where its density becomes equal to that of the ether. Another limit might possibly suggest itself, viz., the point where nitrogen freezes, —214° C., and any air existing at such a level might be supposed to be solid. Although, as we shall see later on, there is practically such a limit for the aqueous vapor of the atmosphere, none exists for the other gases. According to Dewar, the maximum tension of gaseous air at 35° Absolute, which is the boiling point of hydrogen, is somewhat less than .002 mm. of mercury. Of course, at this temperature and at ordinary pressures air is frozen solid, but a portion can always exist in the gaseous state having a certain maximum pressure for each temperature. A slight increase in pressure will condense some of the gas, and this saturation temperature of a gas is called the Dew Point. Thus at —32° C. aqueous vapor has a maximum pressure of .32 mm. of mercury. The pressure at all points in the atmosphere, however, is very much less than the maximum pressure corresponding to the temperature; consequently, the ordinary gases can never become solid or liquid in the atmosphere.

Owing to centrifugal force, due to the earth's rotation, a fluid envelope must assume the form of a spheroid. A surface of water covering the earth would have an equatorial radius greater than the polar radius by something over thirteen miles, or about the ellipticity of the earth. The surface of the atmosphere where it touches the earth has, of course, a like ellipticity, but that of the upper surface is greater. It would be easy to calculate it provided the temperature were constant throughout. But the temperature being less at the poles than at the equator, the ellipticity of the upper surface is somewhat greater than with a uniform temperature. Actually the upper surface of the atmosphere, although approximating to a spheroid, is far from being a mathematically smooth surface, being in fact rough and irregular. We have probably in the sun an analogue which the earth repeats on a smaller scale, viz., protuberances, hollows and vortices. The great eruption of Krakatoa, of which we shall have more to say later on, probably threw up a column of air directly over the volcano far beyond the limits of the atmosphere, most of which was lost. Laplace, in the *Mécanique Céleste,* has shown from certain considerations where the air is supposed to extend to very great distances from the earth, that the ratio of the polar to the equatorial diameter of the upper surface cannot be less than two-thirds. As a matter of fact, since the atmosphere does not extend to any great distance, the ratio is much greater than this, not differing greatly from that of the earth.

In determining the height of the atmosphere we shall neglect the aqueous vapor. A cubic meter of dry air at 0° C. and a pressure of 760 mms. of mercury, that is under standard conditions, weighs 1293.233 grammes at Paris (Regnault's determination). A cubic meter of aqueous vapor under the same conditions weighs five-eighths as much. No clouds have ever been

seen above seven miles from the earth's surface and above twelve miles practically no vapor exists. Since the total amount of vapor in the atmosphere is but a small fraction of its total mass and this small fraction lies nearly all close to the earth's surface, we shall make no very appreciable error in taking the atmosphere as practically dry.

Since the total height of the atmosphere is small compared with the earth's radius, we shall suppose gravity to be constant throughout, thereby introducing only a very small error. We shall further suppose that within the limits of the atmosphere we can neglect the curvature of the earth. Our data are then as follows.

At the bottom of the atmosphere, under standard conditions, a cubic meter of air weighs 1293.233 grammes. At the top the pressure and temperature are zero and a cubic metre of air weighs $\frac{1}{3} \times 10^{-8}$ grammes. Let us consider a parallelopipedon of air resting upon a base of one square meter and extending vertically from the surface of the earth to the superior limit. If we suppose the temperature to be constant throughout the pressure at any level will be equal to the weight of the superincumbent mass and the density of the air at this level will be proportional to the pressure. In descending from one level to another the increment of pressure will be equal to the weight of the section between the two levels and is proportional to the height of the section and its density. Hence $\Delta p = \frac{1}{K} p . \Delta h$, where p is the pressure, h the height of the section, and K is some constant to be determined from the data. Since these values change continuously, we may write $dp = \frac{1}{K} p . dh$ or $dh = K \frac{dp}{p}$. Integrating we have $h = K \log . \frac{p_o}{p_x}$, where p_o is the pressure at the lower level, p_x that of the upper level. Since the density is proportional to the pressure when the temperature remains constant, we may also write $h = K \log . \frac{D_o}{D_x}$. If we take the height of our section as one meter, then $K = \frac{1}{\log . \frac{p_o}{p_1}}$, from which we can determine the value of the constant K. If the weight of a cubic meter of dry air at the given temperature and pressure is w, we have $K = \frac{1}{\log . \frac{p_1 + w}{p_1}}$.

At the surface of the earth, under normal conditions, the pressure is 10,332,790 grammes on every square meter and the weight of a cubic meter of air is 1293.233 grammes. Consequently $K = \frac{1}{\log . \left(1 + \frac{1293.233}{10332790}\right)}$

$$= \frac{1}{.00054318} = 18,409.$$

K is called the barometrical coefficient. It is constant only provided the temperature remains uniform. In fact K is a function of the temperature, and is nearly proportional to the absolute temperature.

Since $w = 1293.233 \times \frac{p_1}{10332790} \times \frac{273}{\tau}$, where τ is the absolute temperature,

$$K = \frac{1}{\log.\left(1 + \frac{1293.233}{10332790} \times \frac{273}{\tau}\right)} = \frac{1}{\log.\left(1 + .00012516 \times \frac{273}{\tau}\right)}$$

$$= \frac{1}{\log.\left(1 + \frac{.0341523}{\tau}\right)}.$$ From this formula we can calculate the value of K for any temperature.

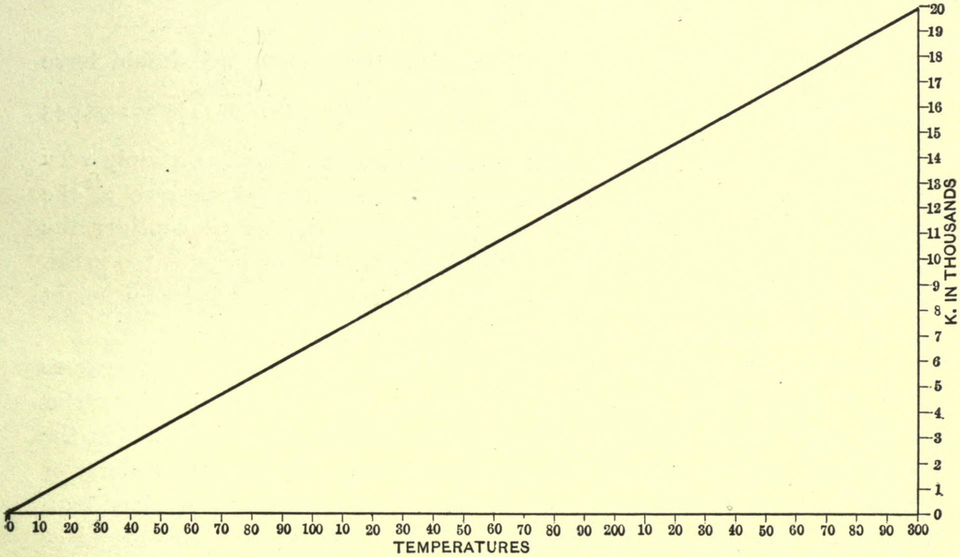

FIG. 2

From Fig. 2 we see that K is practically proportional to the temperature, the curve being nearly a straight line. The accompanying table gives a few values of K for various absolute temperatures.

We shall now attempt the determination of the height of the atmosphere at the equator and at the pole under average conditions, on the supposition that it is at rest relatively to the earth. The average temperature at

τ	K
°	
59	3952
83	5586
103	6944
212	14285
247	16666
264	17821
273	18409
278	18868
291	19608
300	20000

the equator is 27° C., and the average pressure 758 mms. The weight of a cubic meter of dry air under these conditions is

$$1293.233 \times \frac{1}{1 + .00367 \times 27} \times \frac{758}{760} \times \frac{32.088}{32.1747},$$

since 32.088 is the average value of gravity at the equator (British units), and 32.1747 its value at Paris, where the standard determinations were made by Regnault. This is nearly 1175 grammes. The pressure per square meter at the equator is 10,305,925 grammes. Whence

$$K = \frac{1}{\log.\left(1 + \frac{1175}{10305925}\right)} = \frac{1}{.00005} = 20000.$$

If the temperature remained constant throughout we should have $h = K \log. \frac{D_o}{D_x} = 20000 \log. \frac{1175}{\frac{1}{3} \times 10^{-8}} = 20000 \log. 3 \times 10^8 \times 1175 = 230{,}943$ meters, or about one hundred and forty-three miles. This is certainly very much too great, since the temperature decreases to absolute zero at the upper surface. As a first approximation, therefore, let us consider the average temperature as the mean of the extremes, although this is too great. This would give us as the average value of K, 10,000, and the total height would be about seventy-two miles.

If we start from the pole, we have as the average initial temperature —18° C. with perhaps the standard pressure. The weight of a cubic meter of air under these conditions is 1410 grammes and $K = 16{,}835$. On the supposition that the temperature remains constant, $h = 195{,}729$ meters, or about one hundred and twenty-one miles. If we consider the average temperature as the mean of the extremes, we have sixty miles as the height of the atmosphere at the pole. We have then as a first approximation, seventy-two miles and sixty miles, for the equator and pole respectively, as limits *within* which the atmosphere must lie.

Our knowledge of the heights of average isothermal surfaces is very limited and derived from a few soundings by registering balloons. Ferrel some years ago gave the annexed table, which is perhaps a rough approximation.

TABLE OF ALTITUDES, PRESSURES AND TEMPERATURE BY FERREL

Temperature Centigrade	Altitude		Pressures Mms. of Mercury
	Kilometers	Miles	
+20	0	0	760
+10	2.5	1.56	565
0	5.0	3.11	416
—10	7.5	4.66	301
20	10.0	6.21	217
30	12.5	7.77	153
40	15.0	9.32	108
50	17.5	10.87	75
60	20.0	12.42	51
80	25.0	15.53	27
100	30.0	18.63	9
120	35.0	21.74	3
—140	40.0	24.85	1

The following table accords more with the latest obtainable data and is calculated for the equator.

Temperatures		Heights		Densities. Grammes per Cu. Meter	*K*
Centigrade	Absolute	Meters	Miles		
27	300	0	0	1209.6	19608
10	283	2415	$1\frac{1}{2}$	910.9	18868
0	273	4933	$3\frac{1}{4}$	669.9	17821
— 18	255	7244	$4\frac{1}{2}$	497	16666
— 34	239	11268	7	285	15179
— 62	211	16097	10	137	

In trying to extend this table beyond the observed data, we have at hand two methods. We can draw a curve connecting the heights and temperatures and then extend it in the manner the curve tends to run—in other words, we can use the method of extrapolation—or we can attempt to find some formula which connects the known values and then predict by this means the unknown values.

Now if we start from the upper surface and call l the depth of any point below this surface, a formula seems to connect the depth with the temperature quite closely. It is this, $\tau = Cl^2$, where τ is the absolute temperature and C some constant to be determined from the data. Measuring our depth by miles, we have from the preceding table $300 = Cl^2$ and $211 = C(l\text{-}10)^2$, whence $C = .0732$ and $l = 64$ miles. This is the total depth of the atmosphere at the equator.

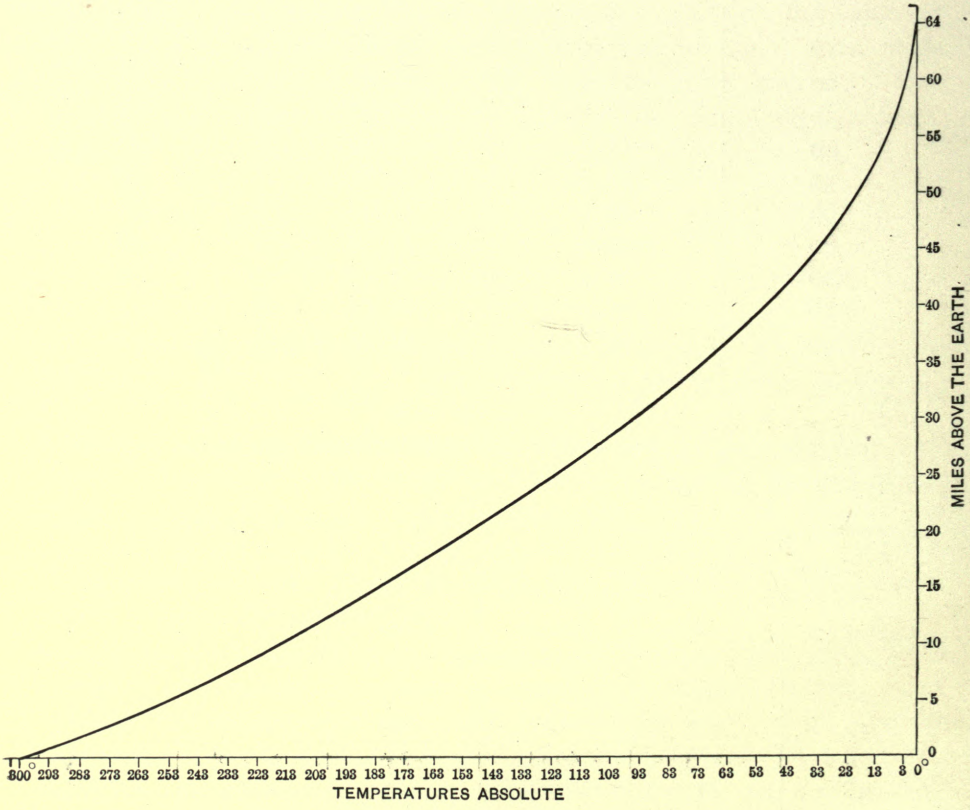

FIG. 3

By using the formula $\tau = .0732\ l^2$, we have for the absolute temperature $211°$, a depth of 54 miles; for the temperature $239°$, a depth of 57 miles; for the temperature $255°$, a depth of 59 miles; for the temperature $273°$, a depth of 61 miles; for the temperature $283°$, a depth of 62.2 miles; and for the temperature $300°$, a depth of 64 miles.

The formula agrees fairly well with the observed facts and is perhaps more accurate than the latter. We can now draw the curve and exemplify

the relation graphically. The curve is a parabola with its vertex at the upper limit.

The height of the atmosphere according to the temperature formula agrees very well with our previous determination. Owing to centrifugal force and because of the lesser temperature at the pole, by the most probable estimate that we can make, the height of the atmosphere at the equator should be about fifteen miles greater than at the pole.

We shall assume, therefore, as the most probable average heights, sixty-seven miles at the equator and fifty-two miles at the pole. The upper surface, therefore, would be a spheroid with an ellipticity of about $\frac{28}{4025}$, while the ellipticity of the lower surface is that of the earth, viz., $\frac{1}{295}$.

We have arrived at the above estimate on the assumption that the atmosphere cannot become less dense than the ether, and from considerations already given we have calculated that the density of the ether is somewhat less than $\frac{1}{3} \times 10^{-14}$ that of water. If we apply in our calculation the estimate of Graetz, viz., that the ether is 10^{-17} the density of water, we shall obtain eighty miles as the average extension of the atmosphere.

While it will always be impossible for us to measure directly the limit of the atmosphere, there are, however, phenomena through which indirectly it has been sought to obtain an estimate. These are shooting stars, auroras, the duration of twilight, and phosphorescent phenomena in the upper atmosphere. The following quotations will serve to give an idea as to what deductions have been drawn from these phenonema.

"Meteorites have been seen at a height of two hundred miles, and as their luminosity is undoubtedly due to friction against the air, there must be air at such a height. This higher estimate is supported by observations made at Rio Janeiro on the twilight arcs by M. Liais, who estimated the height of the atmosphere at between one hundred and ninety-eight and two hundred and twelve miles. The question as to the exact height of the atmosphere must, therefore, be considered as awaiting settlement."—*Ganot's "Physics."*

"But observations on meteors show that the atmosphere really extends to a height of at least one hundred miles, and indeed at that height is sufficiently dense to cause rapid combustion of a meteor passing through it. Observations on the aurora lead at least to a suspicion that this phenomenon sometimes takes place at a height of three hundred, four hundred or five hundred miles. We could hardly suppose it to occur in an absolute vacuum, though it would be unsafe to infer from this that the medium in which it occurs is an extension of the atmosphere proper."—*Professor Newcomb in Universal Cyclopædia.*

"The height of the aurora above the surface of the ground is probably

lower than has been generally stated. Lemström holds that from twenty-two to forty-four miles is a close approximation to the truth; and it may be regarded as certain that even in more southern latitudes the aurora is often seen much lower—at a height of two or three miles, for instance. In polar countries certain forms of aurora, more especially those of weak flames, are seen to proceed from the ground on the tops of certain mountains."—*Ganot's "Physics."*

"A meteor seen over New England on September 6, 1886, and reported by several observers of the New England Meteorological Society, was determined by Professor Newton of Yale College to have become visible at an altitude of ninety miles over northwestern Vermont and to have disappeared at an altitude of twenty-five miles over southeastern New Hampshire."—*W. M. Davis in "Elementary Meteorology."*

"The method of twilight arcs to determine the height of the atmosphere, first devised by Kepler, gives our atmosphere a height of from thirty to thirty-seven miles."—*Flammarion in "The Atmosphere."*

"It is to be noted that different methods give different heights for the atmosphere, but there is no discrepancy, as different things are meant. Thus, if experiments on twilight give forty miles as the height, this implies that the air above this elevation reflects no appreciable amount of light; while if we define the height to be the point where the friction will not set light to a meteor, we have about seventy miles; but, of course, there is no reason why there should not be some air at much greater heights."—*Glaisher.*

"In the year 1798 an investigation of the heights of shooting stars was undertaken by Brandes at Leipzig and by Benzenberg at Düsseldorf. Having selected a base line (about nine miles in length), they placed themselves at its extremities on appointed nights and observed all the shooting stars which appeared, tracing their courses through the heavens on a celestial map, and noting the instants of their appearances and extinctions by chronometers previously compared. Similar sets of observations which have since been repeated lead to the conclusion that the heights of the shooting stars above the ground vary from six to five hundred and fifty English miles, that they move with a velocity of between eighteen and two hundred and twenty miles a second, and that their trajections are frequently not straight lines. It may be asked, how the evolution of light is possible at altitudes so far surpassing the probable bounds of the atmosphere, or where, supposing air to exist, it must necessarily be so attenuated as to approach the limits of absolute vacuity? Poisson, the eminent French geometer, has endeavored to solve the question by affirming the probability of an atmosphere of electricity surrounding the earth and lying above the atmospheric air."—*Hartwig, 1893, "The Aerial World."*

"After the great eruption of Krakatoa, in 1883, the brilliant sunset glows and the longer twilight showed that the dust emitted by the eruption remained

for more than a year suspended at a height of at least sixty miles. The so-called "luminous clouds" seen at night during the same period, and which was probably the same dust still illumined by the sun,* were found by trigonometrical measurements to have about the same altitude. Although it is computed that at a height of seventy miles the air has less than one-millionth of its density at sea level, it is there sufficiently dense to render meteors luminous by friction."—*A. L. Rotch, "Sounding the Ocean of Air."*

"The height of these meteors has been found from simultaneous trigonometrical measurements sometimes to exceed one hundred miles."—*Rotch.*

"The gases composing the atmosphere probably extend to heights proportional† to their density, viz., oxygen to thirty miles, and nitrogen to thirty-five miles, although water vapor nearly disappears at twelve miles."—*Rotch.*

"From these considerations it is supposed that the atmosphere vanishes, as measured by the barometer, at about thirty-eight miles, and this is about the height indicated by twilight, which is the reflected light of the sun when 18° below the horizon."—*Rotch.*

We might multiply these quotations indefinitely, but the above are sufficient. The results are extremely discordant, often very improbable, and at times impossible. Thus the finding of luminous meteors at a height of five hundred and fifty miles moving with a velocity of two hundred and twenty miles a second is clearly impossible. For astronomical reasons the motion of a meteor relatively to the earth cannot exceed forty miles a second. A little consideration will show that it is impossible for two distant observers to trace accurately the path of a meteor against the stars during its momentary flight. The hosts of small flashes during a clear night would make it almost impossible for them to identify the same object, and communication (telegraphic or telephonic) would be of no assistance. At a hundred miles above the earth, if there were any air, its density would certainly be less than that of the ether, and hence we may be satisfied that no meteor has ever been seen at this height. The method is not one susceptible of accuracy, and hence without any hesitation we may throw aside all computations deduced from the flight of meteors.

The method of twilight arcs is even in theory incapable of indicating the upper surface of the atmosphere, for these layers reflect no light. We can get no reflection of the sunlight from the air in a room unless it contains foreign particles (motes). The principle involved can be seen from Fig. 4.

The horizon at the point *A* is *ADB*. When the sun gets below the horizon some light will still be reflected to *A* from the upper layers. If the

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* More probably phosphorescence.—*Author.*

† Probably "inversely proportional" is meant.—*Author.*

point D is in the highest layer capable of reflecting, then the line DC, in which rays from the sun illumine the point D, and are tangent to the earth, is the limit of reflection. Beyond this point no light can be reflected directly to A. The angle BDC is called the twilight arc. Refraction is neglected, the

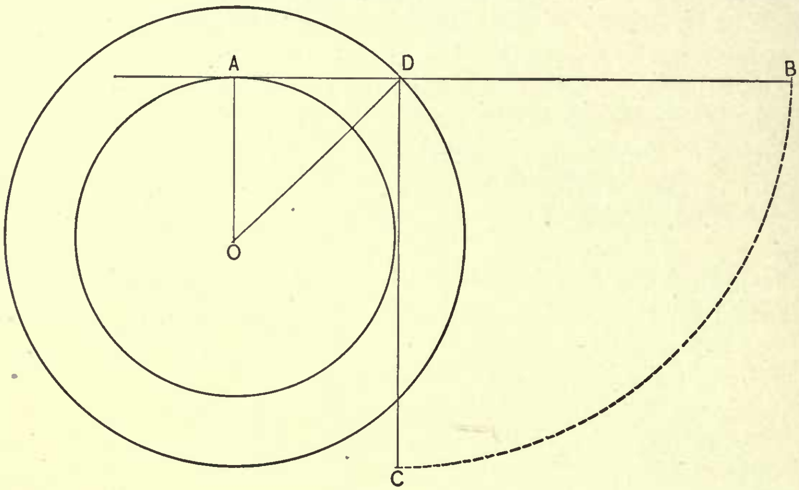

FIG. 4

effect of which would be to increase greatly the twilight arc. The twilight arc is usually about 18°, but it varies with time and place and with the observer. Of course, it is not possible to set a sharp limit to it, since it fades away gradually, and long after the glow has ceased to be perceived, light is still being reflected. The angle ODA is thus 81°, and calling h the height of D above the earth, since the earth's radius is four thousand miles, we have $\frac{R}{R+h} = \frac{4000}{4000+h} = \sin 81 = .98769$. From which we see that $h = 50$. From such a calculation it has been deduced that the height of the atmosphere is fifty miles. But the effect of refraction, as can easily be seen, is to bring down the highest point from which reflections are perceived by a considerable amount. Furthermore, the air itself is phosphorescent. That is to say, that after absorbing the strong radiation of the sun all day it gives out radiation again during the night, both luminous and non-luminous, though with decreasing intensity. It is for this reason that we are able to find our way along roads in the darkest nights even when there are no stars. It is not absolutely dark, owing to this weak diffused illumination from the heavens, but the darkness increases with the lapse of time after the withdrawal of the sun. The old folk saying "It is darkest before the dawn" has something of a scientific foundation. The twilight glow, therefore, fades gradually into this general diffuse illumination of the heavens. From the

above discussion it will be seen that the method of the twilight arc is totally unsuited to the determination of the height of the atmosphere.

The phenomenon of the aurora, if not identical with, is closely analogous to that of the discharge of electricity through gases. As such it must be strictly limited to our atmosphere. In the laboratory it is possible to imitate this glow in an exhausted tube, but when the exhaustion is carried below a certain point all discharge ceases. This limiting density is much above that of the ether. According to Lemström, this glow is never seen at a greater height than forty-four miles, and in all probability this is not far from the truth.

We have seen that neither by the phenomenon of the aurora, nor of meteors, nor yet of the twilight arc, have we a suitable means of determining the height of the atmosphere. There yet remains the phenomenon of the luminous clouds which were seen for a long time after the eruption of Krakatoa and which we have already mentioned. As a result of the explosion the dust and gases from the volcano were blown clear through the atmosphere and a part of them lost in space. There were thus extraneous particles (dust and frozen gases) introduced into the higher layers clear up to the limiting surface. These particles maintained their levels for a long time, sinking very slowly. Trigonometrical measurements were taken of these clouds, with sufficient bases and under conditions favoring extreme accuracy. They were found to be at least sixty miles above the earth, and the limit of luminosity probably coincided very nearly with the limit of the atmosphere. This is practically identical with our estimate of sixty-seven miles. That is to say, we have positive evidence that the atmosphere extends to at least sixty miles at the equator, and as shortly beyond this height it acquires a density equal to that of the ether and the temperature of space, we shall in all probability be quite close to the truth in assigning it such a height. The most probable height, therefore, seems to be about sixty-seven miles at the equator, and fifty-two miles at the pole, and we shall assume these values as the height of the atmosphere.

CONVECTIVE EQUILIBRIUM

We shall next take up the subject of convective equilibrium. We can suppose an ideal case in which the air is heated by conduction from the ground only, and rises through the cold upper strata. It afterwards neither receives nor loses any heat, either by radiation or conduction. As it rises the pressure decreases and consequently it expands and becomes cooler. After the process has gone on for a sufficient time a condition of convective equilibrium will be established; where if we move a portion of the air from one point to any other, it will remain indifferently in equilibrium, having no tendency to change its level.

Since $dp = -dh.D$, where D is the density, and by the adiabatic law $p = p_0 \frac{D^k}{D_o^k}$, where p_0 and D_0 are the pressure and density at the surface of the earth, and k is the ratio of the two specific heats, we have $dp = \frac{p_0}{D_o^k} k D^{k-1} dD = -dh.D$.

Whence,
$$h = \frac{p_0 k}{D_o^k (k-1)} \left(D_o^{k-1} - D^{k-1}\right).$$

Since under normal conditions $p_0 = 10{,}332{,}790$ grammes per square meter, and the density of a cubic meter of air is 1293 grammes approximately, we find that $\frac{p_0 k}{D_o^k (k-1)} = 1456.451$, the value of k being 1.41. Since the density becomes zero at the upper limit, the total height will be $1456.451 \times 1293^{.41} = 27{,}482$ meters, or seventeen miles, which we know to be far from the truth. Since by the thermodynamic law for adiabatic changes, $\tau = \tau_0 \left(\frac{D}{D_o}\right)^{k-1}$, and since the depth or distance from the upper surface is $\frac{p_0 k}{D_o^k (k-1)} D^{k-1}$, it follows that the absolute temperature is proportional to the depth. Hence, under this ideal condition, which is represented by the formula $h = C\,(D_o^{k-1} - D^{k-1})$ the barometric coefficient C remains constant throughout and the temperature is proportional to the depth. The density and the temperature are both zero at the upper limit. If under such conditions a mass of air were introduced of a higher temperature than the level at which it was placed, it would continue to rise to the top, and conversely any air colder than its level would sink to the bottom.

Of course, such an adiabatic condition could not exist in the atmosphere, since the air gains and loses heat at all levels, both by radiation and conduction. The atmosphere at all levels is considerably warmer than would be the case under adiabatic conditions; and consequently a mass of air warmer than its surroundings quickly reaches a level where the temperature and pressure, and hence the density, are the same. As during this short assent not much heat is lost either by conduction or radiation, the mass approximates somewhat to an adiabatic rate of cooling.

We have seen that the atmosphere lies in a comparatively thin sheet over the earth, its average height being about sixty miles. The level of half mass is at 17,400 ft., or half the total mass of the atmosphere lies within this istance of the earth.

VARIOUS ATMOSPHERES

We have seen that the height of the atmosphere may be derived to a first approximation from the formula $H = \frac{K_o}{2} \log. \frac{D_o}{D_u}$, where K_o, D_o are

the barometric coefficient and density at the bottom, and D_u is the density at the top.

Or, $H = \dfrac{\log.\ (w \times 3 \times 10^8)}{\log.\ \left(\dfrac{W+w}{W}\right)^2}$, where w is the weight of unit volume at the bottom, and W is the weight of the total mass of the atmosphere.

Since w, the density at the bottom, is proportional to W, the denominator of the expression above does not change value with the mass of a gaseous envelope surrounding an attracting body. In other words, if we poured a certain mass of a simple gas around an attracting body, or twice the mass or ten times the mass, the barometric coefficient would in all cases remain the same. But w, where W remains the same, would change for different simple gases, since it is the specific density. It is evident then from the expression above that the height of the atmosphere due to a simple gas increases as the mass of the gas increases, but it increases very slowly with the mass. On the other hand, for the same mass of two different gases, the atmosphere of the less dense gas will be higher than that of the other, and the difference of the heights will increase rapidly with the difference of the densities. The barometric coefficient for oxygen at 0° C. is 16,666, and for nitrogen at the same temperature 18,518. An atmosphere of oxygen of the same mass as the atmosphere, and having a uniform temperature of 0° C. at the surface of the earth, would rise to a height of sixty miles, while an atmosphere of oxygen consisting of only one-fifth of this mass, would, under the same conditions, rise to a height of 56½ miles. An atmosphere of nitrogen of the same mass as the air and at 0° C. at the bottom, would extend to sixty-seven miles, while four-fifths of this mass would extend to sixty-six miles. Now, according to Dalton's law, each gas in the atmosphere forms a separate atmosphere just as if it existed alone. In forming a mixed atmosphere, therefore, by taking masses of nitrogen and oxygen in the ratio of 4 to 1, as is the case with air, we should have an atmosphere of nitrogen extending sixty-six miles, and an independent atmosphere of oxygen extending 56½ miles. Consequently, there would be practically no oxygen in the upper limits, and the proportion of oxygen to nitrogen would decrease slightly with the height. Further, the dissipation of the atmosphere at the upper level is practically at the expense of the nitrogen, and the atmosphere becomes proportionally richer in oxygen continually.

Measurements seem to show that there is a decrease in the proportion of oxygen with the height, though within a few miles of the earth this is scarcely perceptible.

AQUEOUS VAPOR

The aqueous vapor in the atmosphere cannot in any sense be considered as a separate atmosphere. Assuming the law $\tau = Cl^2$, we find that the tem-

perature of the atmosphere falls much more rapidly than the pressure. Without entering into the calculation, we may say that, near the surface of the earth, for every meter that we ascend the temperature falls $\frac{578}{112,683}$ of a degree Centigrade, while the fall in pressure is only $\frac{14}{190,330}$. This is a much

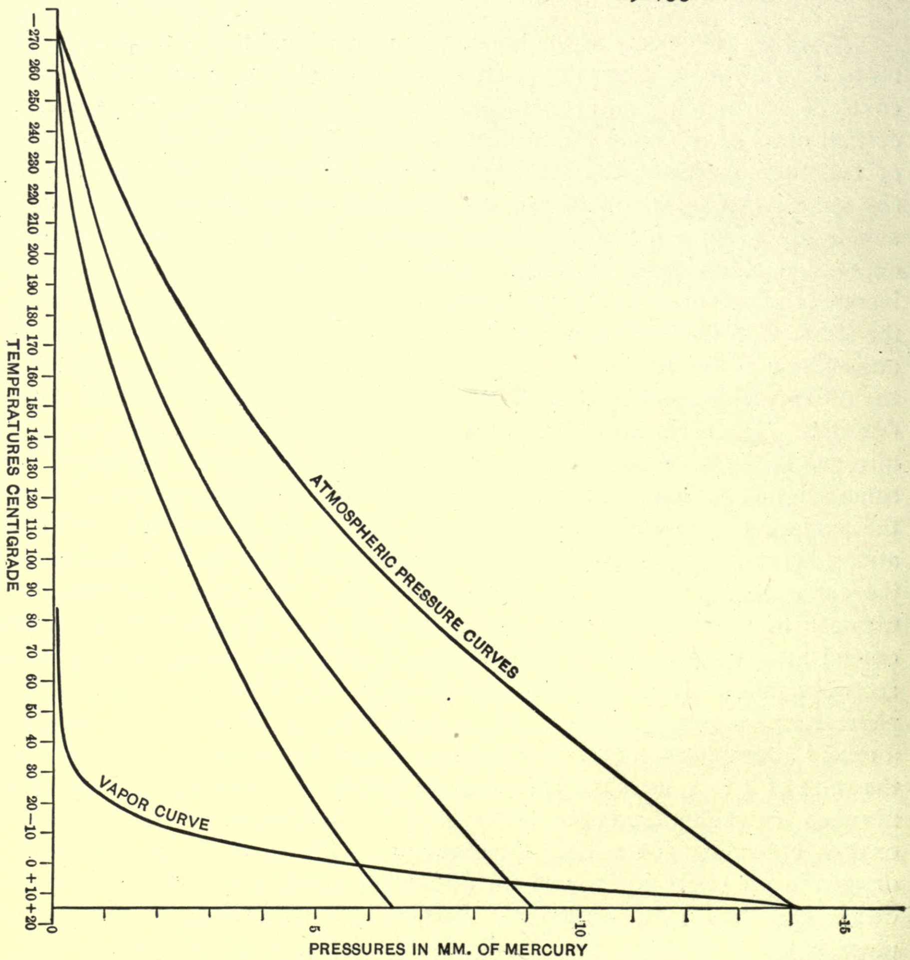

FIG. 5

more rapid rate than the fall of temperature for maximum pressure of vapor at ordinary temperatures.

In the accompanying Fig. 5 we have represented the average fall of pressure with the temperature, starting from different pressures of the magnitude of vapor pressures, as it occurs in the atmosphere. The vapor curve, sometimes called the "steam line," shows the maximum tension of vapor

possible for the corresponding temperatures. It is, in fact, the saturation curve.

The base line represents a temperature of 16° C., the average temperature of the earth's surface. The vapor* curve practically meets the axis of temperature at —80° C., although theoretically some vapor exists even at —273°. If, now, we carry a mass of vapor from the base temperature, 16° C., along one of the pressure curves, it will be subsaturated while it is within the vapor curve. On crossing the vapor curve it is exactly saturated, while above the vapor curve condensation results. Consequently a mass of vapor, endeavoring to arrange itself as an atmosphere about the earth, would be successful until it met the vapor curve, when there would be a sharp discontinuity. If we started with the maximum tension, the discontinuity would occur at once. At a height of ten miles not more than one-fifth gramme per cubic meter could exist as vapor, and the amount decreases rapidly with the height. Theoretically, a trace of vapor should exist even at the upper limit, but it is impossible to detect any by means of instruments at a comparatively short distance above the earth.

PRESSURES IN MM. OF MERCURY AT SATURATION FOR DIFFERENT TEMPERATURES OF AQUEOUS VAPOR

Temperature Centigrade	Force of Vapor, Mms.	Temperature Centigrade	Force of Vapor, Mms.
—32°	0.32	10°	9.17
—20°	.93	15°	12.70
—10°	2.09	20°	17.39
—5°	3.11	25°	23.55
0°	4.60	30°	31.55
+5°	6.53	40°	54.91

Glaisher states that he found the relative humidity to increase slightly up to half a mile, after which it decreased, while at five miles he could detect no vapor. Rotch states that water vapor practically disappears at twelve miles. For an adiabatic expansion of the vapor, the heights at which it could exist would be considerably lessened.

The average amount of vapor in the atmosphere has not been determined. While enormous in the aggregate, still compared to the other constituents it is not much. Practically all of it is within a very short distance

* The vapor curve is very closely represented by the empirical formula, due to Rankine, $p = C(\vartheta - 233)^5$, for ordinary temperatures. For very low temperatures it does not apply.

from the earth, and most of this is in the tropics and the temperate zones. There is comparatively little in the polar tracts. It is most irregularly distributed, great islands and continents of it floating about in some regions, while in others there are gaps or it is practically wanting. It is always undergoing a ceaseless round of condensation and evaporation. Owing to the slowness of the process of diffusion, a mass of vapor may remain isolated for long periods, and a high mountain range is often sufficient to fence it off effectually from vast tracts.

The specific gravity of vapor relatively to air is five-eighths; consequently warm moist air, which is capable of holding more vapor than colder air, is lighter than dry air and has its ascensional power still further increased by its vaporous constituent. Vapor is preëminently the dynamical element of the atmosphere. From its instability it is the chief factor in furthering the more violent phenomena, such as cyclones, tornadoes, thunderstorms, etc.

PRESSURE

The pressure of the air at any point is measured by the barometer. If the atmosphere were perfectly still the pressure would be some function of the depth below the upper level surface. But since the atmosphere is in ceaseless movement, on the fundamental statical element there are superposed transient and mostly slight dynamical elements. Thus, strictly, the barometer is not an indicator of the height or density of the atmosphere overhead alone; it measures the pressure and nothing more. If we expose the face of an aneroid perpendicularly to a violent gust of wind, it immediately shows an increase of pressure, and if we turn its back to the gust there will be a decrease of pressure, while, of course, the total height and density of the atmosphere have not changed. The pressure of the wind against a squarely opposing surface is roughly proportional to the surface area and the square of the velocity of the wind. We can see why this should be so since the impact of the wind is proportional to the mass times the velocity, and the mass or amount of air striking in a given time is proportional to the velocity; hence, the total force is proportional to the square of the velocity. An ascending current of air will naturally decrease the pressure below, while a descending current will increase it.

At the equator there is an extremely regular daily oscillation of the pressure. It is a maximum at the equator, both for amplitude and regularity, but is easily discernible through the torrid zones, becoming finally insensible in the temperate zones. At 10 A.M. at the equator the barometer stands highest, at 4 P.M. lowest. Again at 10 P.M. there is another maximum, while at 4 A.M. occurs a second minimum, the amplitude amounting sometimes to three-twenthieths of an inch. So regular is this oscillation in the tropics that it is said to be possible to tell the time of day by the barometer

within a quarter of an hour. Here we have a phenomenon plainly depending upon the sun. The maxima and the minima are spaced at six hour intervals. The low barometers occur when the temperature is highest for the day and when it is lowest. The high barometers occur at a time when the temperature is the mean of the day. The explanation is as follows. In the tropics the heating effects of the sun's rays is chiefly at the surface of the earth, where the lower layers of the atmosphere become very hot. The upper layers are in fact very diathermanous and are little affected. The first effect of the sun's rays, therefore, is an expansion of the lower layers. This expansion is opposed by the inertia of the upper layers and until the tension is relieved, the tension mounts steadily. At 10 A.M. there is an equilibrium between the rate of expansion and the flow upward, and conse-

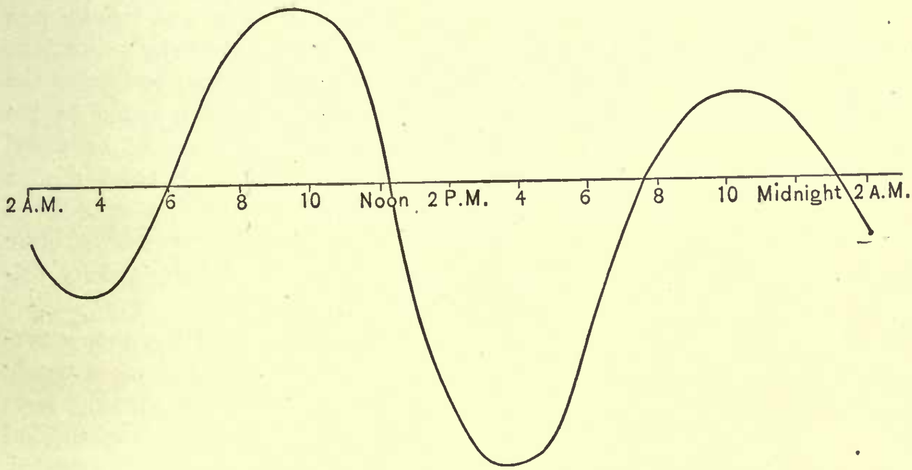

FIG. 6

quently the tension remains momentarily stationary. As the temperature increases the upward flow exceeds the expansion, and the tension falls. This upward movement is a maximum at 4 P.M. when the temperature is greatest. Directly after this the lower layers begin to cool and the upper layers to fall back. This downward movement reaches a maximum at 10 P.M., and at that time the tension becomes a maximum again. After this there is a rebound and the compressed lower layers are relieved by a second upward movement, which reaches its maximum at 4 A.M. Here occurs the second minimum at the coldest part of the day. The greater maximum occurs at 10 A.M. and the greater minimum at 4 P.M. These are the chief ones, the secondary ones being, as it were, mere echoes of the first.

The oscillation is shown graphically by the accompanying diagram. (Fig. 6.) The observations were taken in the Pacific in Lat. 1° S.

The case is somewhat analogous to a man standing on the platform of

a weighing scales and jumping upward. As he jumps the beam goes up; while he is in the air the beam descends; when he falls back on to the platform the beam goes up again; after which the beam falls again and the process repeats itself anew. The air springs from the barometer, sending it up; after the air has acquired its upward momentum the barometer sinks; falling back again on the barometer it sends it up; after which with the cessation of the downward movement it falls again.

Dr. Buchan puts it thus:

"Since the two maxima of daily pressure occur when the temperature is about the mean of the day, and the two minima when it is at its highest and lowest respectively, there is suggested a connection between the daily barometrical oscillations and the daily march of temperature; and similarly a connection with the daily march of the amount of vapor and humidity in the air. The view entertained by many of the causes of the daily oscillations may be thus stated: the forenoon maximum is conceived to be due to the rapidly increasing temperature, and the rapid evaporation owing to the great dryness of the air at this time of the day, and to the increased elasticity of the lowermost stratum of air which results therefrom, until a steady ascending current has set in. As the day advances the vapor becomes more equally diffused upwards through the air, an ascending current, more or less strong and steady, is set in motion, a diminution of elasticity follows, and the pressure falls to the afternoon minimum.

"From this point the temperature declines, a system of descending currents sets in, and the air of the lowermost stratum approaches more nearly the point of saturation, and from the increased elasticity, the pressure rises to the evening maximum. As the deposition of dew proceeds, and the fall of temperature and consequent downward movement of the air are arrested, the elasticity is again diminished and the pressure falls to the morning minimum."

And again: "Some time elapses before the higher expansive force called into play by the increase of temperature can counteract the vertical resistance it meets from the inertia and viscosity of the air. Till this resistance is overcome, the barometer continues to rise, not because the mass of atmosphere overhead is increased, but because a higher temperature has increased the tension or pressure. When the resistance has been overcome, an ascending current of the warm air sets in, the tension begins to be reduced and the barometer falls and continues to fall till the afternoon minimum is reached. Thus the forenoon maximum and the afternoon minimum are simply a temperature effect, the amplitude of oscillation being determined by latitude, the quantity of aqueous vapor overhead, and the sun's place in the sky."

Lord Kelvin* has suggested that the daily oscillation may be considered

* Proceedings of the Royal Society of Edinburgh, 1882.

as a tide produced by the increase of temperature resulting from solar heat. The oscillation, therefore, would be a forced wave completing the circuit of the earth in twenty-four hours, just as the attractional tide of the ocean sweeps around in nearly that period. He finds that the free wave in the atmosphere would have a period something like thirteen hours or about half the diurnal period, and that "the free oscillation produced by a relatively small amount of tide-producing force will have an amplitude which is larger for the half-day term than for the whole-day term."

But the barometric oscillation is in no sense a wave with a surface velocity, and, therefore, not a tide; it is simply a vertical oscillation with no lateral components. If we supposed the earth at rest and, by properly screening the sun, produced the same temperatures in the atmosphere on the exposed side, we should have the same vertical oscillations, but no lateral waves or tides.

PRACTICAL BAROMETRY

The determination of a height by the barometer is equivalent to weighing a column of air under varying conditions. If a column of air between two levels is of uniform temperature and perfectly dry, we can easily determine the height from the formula $h = \dfrac{\log. \dfrac{p_1}{p_2}}{\log. \left(1 + \dfrac{w}{p_1}\right)}$ (1), where w is the weight of unit volume of air under the conditions of temperature and pressure at the bottom (or top). The air, however, is never dry and for considerable differences of level is never of uniform temperature. Consequently we shall have to break up our column into columns small enough to be considered reasonably uniform throughout. Then by adding the heights of our sections, we get the total height between the two extreme levels.

Regnault determined the weight of a liter of dry air at Paris under standard conditions as 1.293233 grammes, and the weight of a liter of mercury as 13,596 grammes. Hence the weight of a cubic foot of dry air at Paris under standard conditions is $\dfrac{1.293233}{13596} \times 12$, expressed in inches of mercury.

If the air still remains dry but has a temperature and pressure other than the standard, the weight will be $\dfrac{1.293233}{13596} \times 12 \times \dfrac{p}{p_0} \times \dfrac{1}{1 + .00367\, t'}$, where t is the temperature centigrade. If the air contains aqueous vapor, we can consider it as consisting of two parts, viz., a cubic foot of dry air at temperature t and pressure $p - f$, and a cubic foot of aqueous vapor at temperature t and pressure f, where f is the tension of the aqueous vapor. The weight of the former is $\dfrac{1.293233}{13596} \times 12 \times \dfrac{p-f}{p_0}$

$\times \dfrac{1}{1 + .00367\, t}$, and the weight of the latter $\frac{5}{8} \times \dfrac{1.293233}{13596} \times 12 \times \dfrac{f}{p_o}$ $\times \dfrac{1}{1 + .00367\, t}$. Hence, the weight of a cubic foot of moist air at Paris expressed in inches of mercury is $\dfrac{1.2393233}{13596} \times 12 \times \dfrac{p - \frac{3}{8}f}{29.9212} \times \dfrac{1}{1 + .00367\, t}$ $= \dfrac{p - \frac{3}{8}f}{1 + .00367\, t} \times .0000381477.$

It will be seen that a change in the value of gravity will leave the numerator and the denominator of formula (1) practically unaffected.

The barometric coefficient is $\dfrac{1}{\log. \left(1 + \dfrac{p - \frac{3}{8}f}{1 + .00367\, t} \dfrac{.0000381477}{p}\right)}$.

The following example will serve to illustrate the method. Two points on the side of a hill were by leveling determined to have a difference of height of exactly fifty feet. The barometer at the lower point read 30.2784 and at the upper point 30.224.

Lower point	Upper point
30.2784	30.224
—.085 reduction to 0° C.	—.085 reduction to 0° C.
30.1934	30.139
+.037 correction for capillarity	+.037
30.2304 corrected reading lower point	30.176 corrected reading upper point

$\frac{3}{8}f$ was found to be .091 inch, and $1 + .00367\, t$ was 1.055.
$p_2 - \frac{3}{8}f = 30.176 - .091 = 30.085$.
Log. 30.085 = 1.4783500.
Log. 1.055 = 0.0232525.
Log. .0000381477 = $\bar{5}.5814485$.
$\therefore$ Log. $\dfrac{p - \frac{3}{8}f}{1 + .00367\, t} \times .0000381477 = \bar{3}.0365460.$

This logarithm corresponds to .0010877 inch. This shows that under the conditions a cubic foot of air weighed .00108 inch of mercury.

$$\begin{aligned} p + .001087 &= 30.1771 \\ \log.\ 30.1771 &= 1.4796775 \\ \log.\ 30.176 &= 1.4796617 \\ \hline &\ .0000158 \end{aligned}$$

$$\text{Hence, } K = \frac{1}{.0000158} = 63291.1$$

$$\log.\ p_1 = 1.4804438$$
$$\log.\ p_2 = 1.4796617$$
$$.0007821 \times K = 49.499 \text{ feet.}$$

Such short distances are a severe test for a barometer. Where the heights are greater the proportional error should be less. By no refinement is it possible to measure with a barometer closer than a foot, since, as we have seen, the weight of a cubic foot of air is equivalent to .001 (+ or —) inch of mercury and barometers do not read finer than thousandths of an inch. By careful work it is probable that the error need not exceed five feet in a thousand. By the method of multiple stations, the errors, which are as likely to be above as below, will be apt to eliminate themselves in the summation. The altitude of Mont Blanc was computed by Delcros, from barometric measurements taken by Bravais and Martins, to be 4810 meters, which came strikingly close to the result obtained by a geodetic survey, viz., 4809.6. But it was a mere coincidence, as Delcros' formula was not susceptible of such accuracy, since it wholly neglects the moisture of the air.

MOTION RELATIVE TO THE EARTH

VERTICAL MOTION

BEFORE taking up the subject of the general circulation of the atmosphere, it will be necessary to consider the effect of the rotation of the earth upon motion at or near its surface. We shall first consider the effect of this rotation upon vertical motion. The effect of the rotation of the earth upon vertical motion has little practical bearing upon meteorology, whereas the effect of this rotation upon horizontal motions is of the utmost importance. For the sake of completeness, however, we shall consider the former.

Rotations may be compounded and resolved in the same manner as simple forces. Thus the rotation of the earth about its axis can be resolved into partial rotations about any set of axes, and the partial rotations about these axes, taken usually mutually perpendicular, together make up the equivalent of the original rotation.

Thus, at any point of the earth's surface, we can draw a vertical, which we will call the axis of z, a tangent to a meridian which we will call the axis of x and an axis mutually perpendicular to these, which we will call the axis of y. If we consider the point at rest, the effect of the earth's rotation is precisely the same as if the rotation about the axis of z were $\omega \sin \lambda$, where λ is the latitude and ω the angular velocity of the earth about its axis.

This rotation is in the northern hemisphere counter clockwise; in the southern hemisphere clockwise. The rotation about the axis of x is $\omega \cos \lambda$, and this rotation is clockwise in the northern hemisphere. There is, of course, no rotation about the axis of y for the same point. We can easily see why this is so by placing a system of axes such as we have described on a globe, and then by moving it along a parallel of latitude to the eastward, noting how, to keep it in the proper position, it is necessary to keep turning the vertical axis in a counterclockwise direction, while the axis of x must be turned clockwise. At the equator we turn only the axis of x, since here $\sin \lambda = 0$ and $\cos \lambda = 1$, while at the pole we turn only the axis of z, since here $\sin \lambda = 1$ and $\cos \lambda = 0$.

Using the present system of coördinates, let us suppose that we drop a particle from a height h, and reckoning the point directly under it as the origin consider y positive when measured to the westward. If the earth were at rest it would strike at the point directly beneath; but actually the

earth is revolving about the axis of x with a clockwise rotation equal to $\omega \cos \lambda$.

Let us first suppose that the direction of gravity does not change during the fall. At starting the body had a velocity relatively to o, the origin of coördinates, of $h \omega \cos \lambda$.

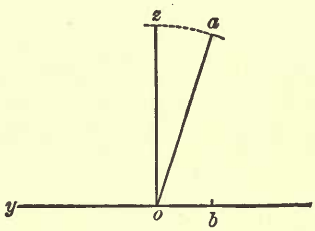

FIG. 7

If, during the time of the fall, the body would have described a small arc $z\,a$, then on striking it would be at a distance $o\,b$ from o equal to the arc $z\,a$. But the direction of gravity is continually changing. The rotation about $o\,z$ does not change the direction, but the rotation about $o\,x$ does, and in the time t turns this direction through an angle $\omega \cos \lambda . t$. There is, therefore, superinduced a westward component to gravity, proportional to the time, and equal to $g \sin (\omega \cos \lambda .\ t)$, which, since the angle is small, we can write $g\, \omega \cos \lambda .\ t$.

The horizontal velocity is, therefore, $g \omega \cos \lambda . \frac{t^2}{2} - h\, \omega \cos \lambda$. When the particle reaches the ground, $h = \frac{g\,t^2}{2}$, so that the horizontal velocity is always eastward except just at the instant of striking, when it is zero. The body, therefore, strikes vertically. There is practically no northerly or southerly component, so that the body strikes to the eastward at a distance $g\, \omega \cos \lambda . \frac{t^3}{6} - h\, \omega \cos \lambda .\ t$ or $\frac{1}{3} g\, \omega \cos \lambda .\ t^3$, from the point vertically under it.

[As a matter of fact, there is an extremely slight motion to the southward, owing to the rotation of our system of axes about the axis of z in a counterclockwise direction. Careful experiments show that objects dropped from a height show a constant easting which agrees well with theory. The deviation for one hundred feet is only about one-eleventh of an inch, but by repeated droppings such a small quantity can be brought out. The southing is, of course, too small to be detected.]

If, on the other hand, we suppose a body to rise under a flotative force, such as a balloon or a mass of hot air, to a height h, it will have a horizontal velocity $h\, \omega \cos \lambda - f\, \omega \cos \lambda . \frac{t^2}{2}$, where f is the ascensional force, supposed to remain constant. Since $h = \frac{1}{2} f .\ t^2$, the horizontal velocity at all times is zero. In other words, the westerly lag is exactly overcome by the easterly component of ascensional force due to its change of direction, so that the balloon or hot air would rise vertically, or always remain over its starting point, provided the air were perfectly still. J. M. Bacon states that on one

occasion, in perfectly still air, his balloon rose perpendicularly and "remained hovering over the Crystal Palace grounds, and apparently *over the same spot* in the grounds, for some twenty minutes, till, as altitude increased, the whole enclosure had to all appearances shrunk to the dimensions of a toy model."

A falling balloon or a falling mass of air would fall directly to the eastward, and if we suppose the resistance of the air to be proportional to the square of the velocity, or $r = Cv^2$, this deviation to the eastward would be $\frac{1}{3}\ \omega \cos \lambda.\ g t^3 \left(1 - \frac{C g t^2}{4}\right)$.

If a projectile were shot directly upwards to a height h, it would experience besides the westward lag, an acceleration to the westward due to the change of direction of gravity. The relative westward velocity would, therefore, be $h\ \omega \cos \lambda + g\ \omega \cos \lambda\ \frac{t^2}{2}$. Since at the summit $h = \frac{1}{2} g t^2$, where t is the time of ascent or descent, the relative westward velocity at the summit is $g\ \omega \cos \lambda.\ t^2$. The distance to the westward at the summit due solely to the lag is $h\ \omega \cos \lambda\ t$, and that due to the acceleration is $g\ \omega \cos \lambda.\ \frac{t^3}{6}$. Therefore, the total westing at the summit is $\frac{2}{3}\ g\ \omega \cos \lambda.\ t^3$. The projectile, therefore, starts falling at a distance $\frac{2}{3}\ g\ \omega \cos \lambda\ t^3$. to the westward of the vertical, and with a westward velocity of $g\ \omega \cos \lambda\ t^2$.

If it were at rest at the summit, it would gain $\frac{1}{3}\ g\ \omega \cos \lambda\ t^3$ to the eastward on striking, and, therefore, would be this distance to the west. But the westward velocity would carry it $g\ \omega \cos \lambda\ t^3$ to the west of the vertical. Therefore, the total westing is $\frac{4}{3}\ g\ \omega \cos \lambda\ t^3$ or $\frac{4}{3}\ \frac{\omega \cos \lambda\ V^3\ t^3}{g^2}$, where V is the initial velocity upwards. This value is given by Laplace in the "Mécanique Céleste," IV., p. 341.* By the formula given above for descent under resistance the easting is $\frac{1}{3}\ \omega. \cos \lambda.\ g t^3 \left(1 - \frac{C g t^2}{4}\right)$, where t is the time of descent. When the descent is slow this becomes converted into a westing. Hence, ashes ejected from a volcano into still air would, on reaching their highest point, be to the westward of the volcano and have a westward velocity. Even if at rest, they would fall to the westward, and this westing is further increased by the initial westward velocity at the beginning of the fall.

As an example, let us suppose a bullet fired upwards at the equator with an initial velocity of 2,000 ft. per second. The resistance of the air is neglected. The times of ascent and descent are 62.5 seconds, while the height

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* Ferrel gives for the easting when a body is dropped from a height $\frac{2}{3}\ g\ \omega \cos \lambda\ t^3$, and for the westing of a projectile shot upward $\frac{8}{3}\ \frac{V^3\ \omega \cos \lambda\ t^3}{g^2}$ ("On the Motion of Fluid and Solid Bodies Relative to the Earth"). Both are incorrect.

reached is 62,500 ft. At its highest point the bullet is 372.65 ft. to the west of the vertical and 745.3 ft. to the west on striking. The velocity to the westward at the summit is nine feet per second, or more than six miles per hour.

Again, neglecting the resistance of the air, let us suppose that the volcano Krakatoa, during the great eruption of 1883, drove its contents upwards with an initial velocity of 6272 ft. per second, which is about twice the highest velocity attained for projectiles. Some of the particles would reach a height of 116 miles, or twice the height of the atmosphere. At the highest point they would have a westward velocity of 88 ft. per second, or 60 miles per hour. The time of ascent would be 196 seconds. At the highest point they would be 11,498 ft. to the westward, or more than two miles. If we consider the gases merely of the explosive wave, which, as we shall see later, may be propagated with very great velocities, these would be quickly condensed into solid particles (SO_2 and H_2O), which on striking the atmosphere on their return would have their velocities destroyed by the resistance of this medium, although passing upward through the air with the wave they would not experience this resistance.

Stokes has given the following formula for the velocity of particles falling through a viscous medium. $V = \frac{2g}{9\mu^1}\left(\frac{\sigma}{\rho} - 1\right)a^2$, where σ and ρ are the densities of the particles, and the medium respectively, a is the average radius of the particles and μ_1 is what Stokes calls the "Index of Friction." This index of friction is defined by the equation $\mu^1 = \frac{\mu}{\rho}$, where μ is the coefficient of viscosity. μ^1 is taken as $\overline{.116}^2$ for air. From this formula the following table is derived.

Radius of Particles in Inches	Feet per Day	Time of Falling 50,000 Feet
.00003	68	2 years
.00007	368	136 days

As the atmosphere did not become clear for two years after the eruption of Krakatoa, it is inferred that the particles averaged about .00006 inch in diameter, which agrees well with other theoretical deductions based upon the optical phenomona of Coronæ, etc.

The above treatment of relative vertical motion is that of Routh ("Advanced Dynamics"). As it does not take into consideration the fact that the moment of the velocity about the earth's axis remains constant, it is merely an approximation, though when the distances are small a very close one.

According to the formulas above it will be seen that the body when dropped from aloft strikes the earth vertically to the eastward, while when projected upward and falling back again it has a westward velocity on striking.

Strictly, on account of the conservation of areas, the body on falling back after being projected upward must strike vertically, while, when dropped from aloft, it must have an eastward velocity or striking. Still, for short distances, these velocities are so slight that the formulas given above are nearly correct.

A rigorous treatment is as follows. Since the body when projected upward or dropped from aloft always remains in a fixed plane which passes through the center of the earth and is tangent to the parallel of latitude at the initial point, the body in either case falls south of the starting point in the northern hemisphere. If we suppose the body dropped from a height h, so that $R + h = R'$, where R is the radius of the earth, and φ is the angle in the fixed plane between the radius vector, r, at any time and its initial position, then $\dot{\varphi} = \frac{R'^2 \omega \cos \lambda}{r^2}$. This follows from the law of the conservation of areas. We shall throughout adopt the Newtonian notation of a super dot for velocities and a double super dot for accelerations.

Hence, $\dot{\varphi} = \frac{R'^2 \omega \cos \lambda}{\left(R' - \frac{g t^2}{2}\right)^2}$. Calling $\frac{2 R'}{g}$, a^2, we have $\dot{\varphi} = \frac{4 R'^2 \omega \cos \lambda}{g^2} \cdot \frac{1}{(a^2 - t^2)^2}$.

Since $\frac{1}{(a^2 - t^2)^2} = \frac{1}{4 a^2}\left[\frac{1}{(a+t)^2} + \frac{2}{(a+t)(a-t)} + \frac{1}{(a-t)^2}\right]$, we can write $\varphi = \frac{R' \cos \lambda \, \omega}{2 g} \int \left[\frac{1}{(a+t)^2} + \frac{2}{(a+t)(a-t)} + \frac{1}{(a-t)^2}\right] d t$.

$$\text{Or, } \varphi = \frac{R' \omega \cos \lambda}{2 g} \left[\frac{1}{(a-t)} - \frac{1}{(a+t)} - \frac{1}{a} \log. \frac{(a-t)}{(a+t)}\right].$$

Neglecting the small southing, the easting is $R(\varphi - \cos \lambda . \omega t)$. As a practical example let us suppose that a body falls from the height of a mile at the equator. The time is 18 seconds and a is found to be 1143.77. Substituting these values, we get as the value of φ, $\varphi = .0013092048$. Since the radius of the earth at the equator is 20,926,202 ft., $R' = 20,931,482$. $R \varphi = 27,396.68$ ft. The earth has advanced 1521.753 ft. per second or 27,391.56 ft. in the time of fall. Hence, the easting is $R(\varphi - \omega t) = 5.12$ ft. Where the height is considerable and the southing appreciable, we may determine both the easting and southing by solving a spherical triangle. The easting given by the approximation formula of Laplace and Routh, viz., $\frac{1}{3} g \omega \cos \lambda \, t^3$, is

4.52 ft. For lesser heights the values given by the two formulas are practically identical.

HORIZONTAL MOTION

We have now to consider the important problem of the effect of the rotation of the earth upon bodies moving horizontally or parallel to its surface. The motion is supposed to take place without any friction. We shall first consider the hypothetical case that the earth is a sphere and then the actual case of a spheroid. To fix our ideas, we shall suppose a smooth ball rolling over a perfectly smooth sphere of the size of the earth. We must distinguish between the absolute motion of the body and its motion relatively to the earth when the latter is rotating on its axis. It will be seen that it is impossible for the body to maintain a position relatively at rest to the earth, except at the equator. Since there is absolutely no interaction between the body and the earth except in a direction at right angles to the motion, the body being given an initial absolute velocity in a certain fixed plane will always remain in that plane, describing in space a fixed great circle. The absolute velocity will remain constant and the path described on the rotating earth will be a composition of the motion of the earth and the motion of the body in the fixed great circle.

The angular velocity of the earth is .2618 radian per hour. If we suppose the body projected with a velocity of eighty miles an hour due east at 30° Lat., it will describe a fixed great circle tangent to the parallels 30° N. and 30° S., crossing the equator at points equidistant from these two extremes. The velocity of the earth at this latitude is 898 statute miles, since the velocity at the equator is 1037 miles. The absolute velocity of the body is 978 miles.

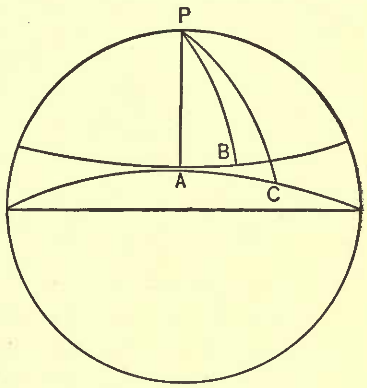

FIG. 8

At the end of an hour the point from which it started on the earth's surface will be 898 miles to the eastward on the same parallel of latitude. The body, however, will be on the great circle AC, at a distance AC equal to 978 miles. The angle $APB = 15°$. The angle $AC = 14°\ 9'$, since we take the radius of the earth to be 3958 miles, and the circumference 24,869 miles. By spherical trigonometry we find that the angle $APC = 16°\ 14'$. A great circle tangent to the parallel of 30° at B is the direction of due east at B. This great circle cuts the circle PC at a point in latitude 29° 39′, while C is in latitude 29°. Consequently the body has deviated to the right 39′ or 39 geographical miles from due east, and is moving in an absolute direction 8° 2′ south of east.

It has, therefore, a component of velocity of 968.4 miles due east and another of 136.7 miles due south. The earth at this point, *C*, has a velocity of 907 miles. Therefore the relative eastward velocity will be 61.4 miles instead of 80 miles, as it was originally. This compounded with the southward velocity of 136.7 miles gives a velocity relatively to the earth of 150 miles, and the direction relatively to the moving earth is 66° south of east. It crosses the equator at an absolute angle of 30°, with an absolute eastward velocity of 847 miles and a southward velocity of 489 miles.

Relatively to the earth, it will have at the equator a *westward* velocity of 190 miles. Hence relatively to the earth it will cross the equator with a velocity of 524 miles and in a direction 68° 46′ south of west. We can easily compute that after two hours from starting the body will be in Lat. 26° 7′ N. and moving with a velocity relatively to the earth of 12 miles to the east.

Shortly after this, just below Lat. 26°, it will have no relative east or west motion.

After three hours, it will be in Lat. 21° 39′ N., and moving with a relative west velocity of 53 miles, and with a southward velocity of 355 miles. The total relative velocity will, therefore, be 478 miles, and it will be moving in a direction 81° 31′ south of west.

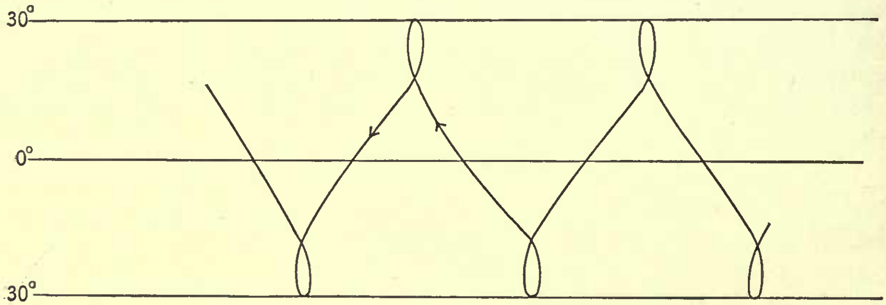

FIG. 9

It will be seen that the body will describe on the moving earth a path like that in Fig. 9, moving gradually to the westward and oscillating between Lats. 30° N. and S., while executing a loop at these limits.

If the body were started without any relative velocity, that is, if we imparted to it an absolute velocity of 898 miles due east at 30°, it would oscillate between the parallels of 30° as before, but the loops would be changed into cusps. If we imparted to it a relative velocity due west, it would trace on the moving earth a wavy curve like in Fig. 10.

As another example let us suppose that the body is started due west at 30° with a relative velocity of 98 miles. Since the earth here has a velocity

to the eastward of 898 miles, the absolute velocity of the body is 800 to the east. At the end of an hour it will be in Lat. 29° 20′, and will be moving in an absolute direction, i.e., a direction referred to stationary coördinates of 6° 34′ south of east. The point from which it started will have moved

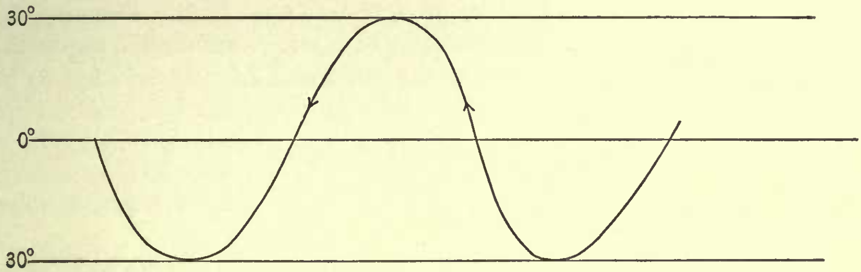

FIG. 10

on the parallel of 30° a distance of 898 miles. The great circle tangent to this point will cut the meridian on which the body now is in Lat. 29° 59′. The direction of west lies on this great circle. Consequently, the body which was projected due west is now 39 geographical miles to the *left* of west.

Consequently, it is not true that a projectile always deviates towards the right in the northern hemisphere.

FRICTIONLESS MOTION OVER A ROTATING SPHEROID

We now come to consider the case of a body moving without friction over a centrally attracting spheroid. The surface of still water on the earth assumes a spheroidal form because it is subjected to two forces, viz., gravity acting downward and centrifugal force acting in a direction at right angles to the axis of the earth and away from it. The resultant of these two forces must be everywhere normal to the surface of still water and by Clairaut's Theorem such a level fluid surface is a spheroid with the minor axis through the poles. At the equator the centrifugal force is about $\frac{1}{290}$ of the true force of gravity, so that bodies there are lighter by this amount. The weight of a body at any other point is its true gravitation diminished by the component of the centrifugal force in the direction of gravity.

We shall suppose, as before, a small sphere rolling upon a perfectly smooth spheroid which represents the earth. It will now be possible for the body to remain at rest relatively to the earth. All that is necessary is that it shall have the same velocity as the point on which it rests. There is now an equilibrium between the gravitational and centrifugal forces, and a very delicate one at that. The slightest difference in the velocities will set the body in motion and in time it will drift far from its original position. We shall designate motion along a meridian as polar, and motion along a

parallel of latitude as horizontal. The latter term is chosen for want of a better one, as it would be inconvenient to call it east-west motion.

If now the body moves due west relatively to the earth, an acceleration will urge it towards the pole, while if the body move due east an acceleration will urge it towards the equator. Let us suppose that the body is moving due west at Lat. 30° with a velocity of eighty miles an hour. Denoting the earth's radius by R, its angular velocity by ω, and the angular velocity of the body about the axis of the earth, or the horizontal angular velocity, by $\dot{\psi}$, we have for the acceleration,

$$f = R \sin \vartheta \cos \vartheta \, (\omega^2 - \dot{\psi}^2),$$

a positive value denoting acceleration towards the pole, a negative value towards the equator.

Since $\omega = .2618$ and $\dot{\psi}$, under these conditions is .2385, we have $\omega^2 = .06853$ and $\dot{\psi}^2 = .0568$, $\omega^2 - \dot{\psi}^2 = .0117$, $\sin \vartheta \cos \vartheta = .433$ and $f = 20$, where the acceleration is expressed in miles per hour. Expressed in foot-seconds it is .0082, which is only $\frac{1}{4000}$ of the acceleration of gravity.

What it amounts to, is that the atmosphere, owing to the rotation of the earth, is whirled away from the pole against gravity, bulging out until it acquires a position where the tangential components of gravity and centrifugal force balance each other. This equilibrium, as we have said, is a very delicate one, and when the proper velocity about the earth's axis is not reached, the air falls again towards the pole. In the case we have just considered of a wind due west with a velocity of eighty miles an hour at 30° Lat., the air is urged back towards the pole at the rate of twenty miles an hour.

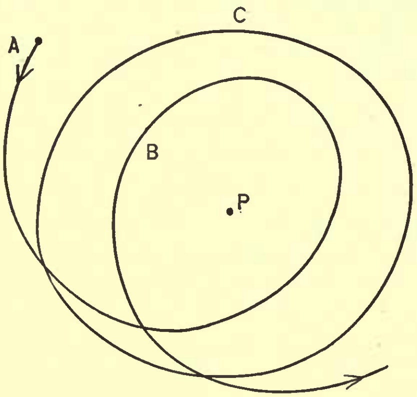

FIG. 11

Since the force is a central one, the areas described by our body about the axis of the earth in equal times must always be equal, or the law of conservation of areas must hold.

Let us suppose we are looking down on the spheroid as in Fig. 11. The pole P is in the center. If now we start our body at A with a horizontal angular velocity, $\dot{\psi}$, less than ω, the angular velocity of the earth, it will be urged poleward and will describe a spiral about the axis represented by the path $A\,B$. By the law of conservation of areas its angular velocity will increase as it gets nearer to the axis and will finally be greater than ω, which remains constant. The body will, therefore, oscillate between two extreme latitudes. That

is, it will at first be urged towards the pole and then its velocity increasing with the latitude, it will be actuated by an acceleration in the reverse direction, which will first overcome its inertia towards the pole and finally drive it back again to the latitude from which it started. At a certain point in its path it will have precisely the horizontal angular velocity of the earth, and at this point will suffer no acceleration in either direction. The latitude where this occurs we have represented by the circle C, and we shall call this circle the circle of equilibrium. The circle of equilibrium may, therefore, be considered the circle of mean position. The work done on the body by the excess force while the body is outside the circle creates an extra velocity which is destroyed by the work done by the repellent force in pushing it out of the circle again.

Calling v_h the horizontal velocity, since the moment of this velocity about the earth's axis remains constant, we have $\cos^2 \vartheta \dot{\psi} = C$, where ϑ is the latitude. Since the poleward acceleration is

$$R \ddot{\vartheta} = R \sin \vartheta \cos \vartheta \left(\omega^2 - \frac{C^2}{\cos^4 \vartheta}\right)$$

Or,
$$\ddot{\vartheta} = \frac{\sin 2 \vartheta}{2} \omega^2 - C^2 \tan \vartheta . \sec^2 \vartheta .$$

Multiplying by $\dot{\vartheta}$ and integrating,

$$\dot{\vartheta} \ddot{\vartheta} = \frac{\sin 2 \vartheta}{2} \omega^2 \dot{\vartheta} - C^2 \tan \vartheta \sec^2 \vartheta \dot{\vartheta} .$$

$$\frac{\dot{\vartheta}^2}{2} = - \frac{\cos 2 \vartheta}{4} \omega^2 - \frac{C^2}{2} \tan^2 \vartheta + K.$$

Or,
$$\dot{\vartheta}^2 = \frac{\cos 2 \vartheta_0 - \cos 2 \vartheta}{2} \omega^2 + C^2 (\tan^2 \vartheta_0 - \tan^2 \vartheta) \qquad (1).$$

Since $\cos^2 \vartheta = \frac{C}{\dot{\psi}}$, $\cos 2 \vartheta_0 - \cos 2 \vartheta = \frac{2 C}{\dot{\psi}_0} - \frac{2 C}{\dot{\psi}}$

and $\tan^2 \vartheta_0 - \tan^2 \vartheta = \frac{1}{C} (\dot{\psi}_0 - \dot{\psi})$.

We can therefore write Equation (1)

$$\dot{\vartheta}^2 = C (\dot{\psi}_0 - \dot{\psi}) \left(1 - \frac{\omega^2}{\dot{\psi}_0 \dot{\psi}}\right) \qquad (2)$$

Two values of $\dot{\psi}$ make $\dot{\vartheta}$ vanish, viz., $\dot{\psi} = \dot{\psi}_0$, and $\dot{\psi} = \frac{\omega^2}{\dot{\psi}_0}$. The latter value is the extreme upper value of $\dot{\psi}$, which we shall designate by $\dot{\psi}_u$. It is thus seen that the angular velocity of the earth is the geometric

mean between the greatest and least horizontal angular velocities of the body, corresponding to the highest and lowest latitudes which it reaches.

From (2) we have,

$$v_p^2 = \frac{1}{v_h^2}(v_h^2 - v_o^2)(v_u^2 - v_h^2) \qquad (3),$$

where v_p is the polar velocity and v_u and v_o represent the extreme horizontal velocities, while v_h is the horizontal velocity at any intermediate point.

It will be seen that the maximum polar velocity occurs when the horizontal velocity is a geometric mean between the two extreme horizontal velocities or when $v_h = \sqrt{v_u v_o}$. Writing v_{pm} for the maximum polar velocity, we have

$$v_{pm} = v_u - v_o \qquad (4).$$

The horizontal angular velocity at this point is likewise the geometric mean of the two extreme horizontal angular velocities, or ω.

This occurs on the parallel of equilibrium. Here, relatively to the earth, the horizontal velocity is zero, so that the path traced by the body on the moving earth is first westward, then curves round towards the right, cutting the parallel of equilibrium at right angles. It continues to curve to the right until it goes due east, where it reaches its highest latitude. After this it cuts the parallel of equilibrium at right angles going due south, and turns due west again at its lowest latitude.

Since $\cos^2 \vartheta_o \dot{\psi}_o = \cos^2 \vartheta_u \dot{\psi}_u$ and $\dot{\psi}_u = \frac{\omega^2}{\dot{\psi}_o}$, we have $\cos \vartheta_o \dot{\psi}_o = \cos \vartheta_u \, \omega$ (5), or the absolute velocity at the lower latitude is equal to the velocity of the earth at the upper latitude. In the same way we can show that the absolute velocity of the body at the upper latitude is equal to the velocity of the earth at the lower latitude.

Thus Equation (5) can be written:

$v_o = v_{eu}$ (6) or $v_u = v_{eo}$ (7), where v_{eo} and v_{eu} are the velocities of the earth at the lower and upper limits respectively.

From these equations or from Equation (1) the values for one of the limits can be determined when those of the other are known.

It will readily be seen that the path of the body as traced upon the moving earth will be like that represented in Fig. 12. The body will continually oscillate between two extreme parallels of latitude, cutting the parallel of equilibrium at right angles and moving continually towards the west while executing a series of symmetrical loops.

As an example, let us suppose the body launched due west at 10° N. Lat. with a velocity relatively to the earth of 226 miles an hour. The absolute initial velocity is 1021 — 226 = 795.

The initial horizontal angular velocity is therefore .204 and $\omega = .2618$. Since $\dot{\psi} = .204 = \frac{C}{\cos^2 \vartheta}$, we find that $C = .19785$. $C^2 = .03914$ and $\omega^2 = .06853$. We find that the superior latitude of the spiral will be 39° 57′, as the earth moves here with a velocity of 795 miles. (By Equations (6) and (7).)

The circle of equilibrium, where there is no polar acceleration and the maximum polar velocity, is 28° 53′. Here the polar velocity is 226 miles

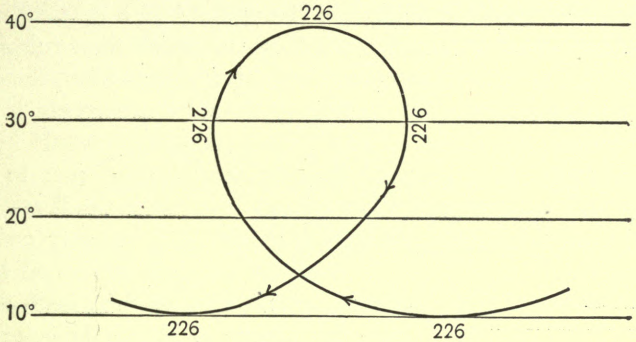

FIG. 12

and the horizontal velocity 908, the same as that of the earth. At the upper limit, 39° 57′, the horizontal angular velocity is .3359 and the horizontal velocity 1021 miles. Hence relatively to the earth it is moving here with an eastward velocity of 226 miles. The absolute velocity at the parallel of equilibrium is 929 miles. Relatively to the earth the velocity is 226 miles at all points.

If the body had been launched at 10° with a velocity of eighty miles to the west, the superior limit would be about 25° N. Lat. and the parallel of equilibrium would be 18° 55′. If we should launch it to the east at 10° instead of to the west, there are three cases. With velocities above sixteen miles, it would describe a path like Fig. 13, oscillating between 10° Lat. N. and S., and moving gradually east or west according to its velocity.

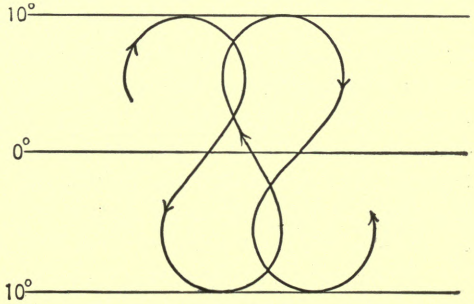

FIG. 13

With velocities less than sixteen miles an hour it would remain wholly in the northern hemisphere and describe a path like the first one figured. But if launched with a velocity of exactly sixteen miles it would curve

around onto the equator, where it would remain, proceeding west with a velocity of sixteen miles.

The above discussion of the equilibrium of a body on a rotating spheroid where there are no frictional forces applies directly to the equilibrium of fluids on such a rotating spheroid. Supposing the fluid to be throughout of a uniform temperature, it can only be at rest relatively to the spheroid under the conditions that the surface of the fluid assumes the form of an ellipsoid of revolution and that each particle has its proper velocity which is a function of its distance from the axis. So delicate is this equilibrium that, as we have seen, a comparatively slight change in the velocity of a particle will cause it to depart widely from its original position, to which as a rule it never returns. If, then, we were to heat such a fluid in equilibrium, the atmosphere for instance, unequally, thereby setting up convection currents, it will be seen that these currents would differ widely from what they would be on a non-rotating spheroid. The unequal heating would give rise to convective velocity, but this velocity would immediately disturb the delicate equilibrium and start the particles upon paths such as we have recently considered.

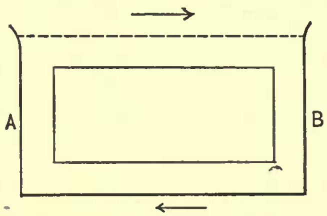

FIG. 14

If we have a fluid contained in a vertical circuit, with a free surface, such as *A B,* and heat the column *A* above the column *B,* since all fluids expand under the influence of heat, the column *A* will become higher than *B,* and under the influence of gravity the top of the column will fall (flow) towards *B.* There will thus be set up a circulation of the fluid from *A* to *B* in the direction of the arrows, which will continue as long as there is a difference of temperature between the two ends. We shall thus have a heat engine which performs the work of setting the fluid in motion and overcoming the friction. When the heat energy used up is exactly equal to the work of friction, the motion will be uniform. Since a hot column of air weighs less than a cold column of equal height, the pressure at the bottom of *A* must be less than at the bottom of *B.* It is thus that differences of temperature cause differences of pressure in a fluid and hence motion.

If we supposed the earth at rest and the surface everywhere at its actual temperature, then there would be a flowing away of the air from the equator at its upper surface towards the poles, and a flowing back along the lower surface towards the equator again. Since the mass in circulation is constant, and the available space about the poles is constricted, the velocity would have to increase with the latitude, so that the same volume might cross different sections in the same time.

We have already seen that from the law of conservation of areas $\psi = C \sec^2 \vartheta$.

Hence, $\ddot{\psi} = 2\, C \sec.^2 \vartheta \tan \vartheta\, \dot{\vartheta} = 2\, \dot{\psi} \tan \vartheta\, \dot{\vartheta}$ and
$$R \cos \vartheta\, \ddot{\psi} = 2\, R\, \dot{\psi} \sin \vartheta\, \dot{\vartheta}.$$

Putting $\dot{\psi}_r$ for the horizontal angular velocity relatively to the earth, or $\dot{\psi}_r = \dot{\psi} - \omega$, we have

$$R \cos \vartheta\, \ddot{\psi} = 2\, R\, (\dot{\psi}_r + \omega) \sin \vartheta\, \dot{\vartheta} \text{ and } R\, \ddot{\vartheta}$$
$$= - R \sin \vartheta \cos \vartheta\, (2\, \omega\, \dot{\psi}_r + \dot{\psi}_r^2).$$

Since $\dot{\psi}_r$ is usually small compared with ω, we may without serious error write these two equations,

$$R \cos \vartheta\, \ddot{\psi} = 2\, R\, \omega \sin \vartheta\, \dot{\vartheta} \text{ and } R\, \ddot{\vartheta} = -\, 2\, R \sin \vartheta \cos \vartheta\, \omega\, \dot{\psi}_r.$$

The square root of the sum of the squares of these quantities is the deflective force relatively to the earth. Consequently, if ρ be the radius of curvature of the path of the body and v_r the relative velocity (total) at any point,

$$R \sqrt{4\, \omega^2 \sin^2 \vartheta\, \dot{\vartheta}^2 + 4 \sin^2 \vartheta \cos^2 \vartheta\, \omega^2\, \dot{\psi}_r^2} = \frac{v_r^2}{\rho}.$$
$$= 2\, \omega \sin \vartheta \sqrt{R^2 \dot{\vartheta}^2 + R^2 \cos^2 \vartheta\, \dot{\psi}_r^2} = 2\, \omega \sin \vartheta\, v_r.$$

Hence, approximately, $\rho = \dfrac{v_r}{2\, \omega \sin \vartheta}$ and the angular velocity with which the path turns is $2\, \omega \sin \vartheta$.

This result was first given by Ferrel, but its application seems to have been generally misunderstood by meteorologists, as the following example of what is found in some text-books, shows.

"*If a body be supposed to move without friction on the level surface of a rotating globe, a single impulse would give it perpetual motion; but the motion could not be along a straight path. It would continually be deflected to one side of its momentary path, to the right in one hemisphere, to the left in the other, and with a force dependent on its velocity and on its latitude; but independent of its direction of motion. Its path would always be curved in a systematic manner: the curvature would be sharper for slow motions than for rapid motions, and sharper in high latitudes than near the equator. The following table will give some idea of the rate at which a moving body tends to turn from a straight line on a sphere rotating once in twenty-four hours.

RADIUS OF CURVATURE (IN MILES) FOR FRICTIONLESS MOTION ON THE EARTH'S SURFACE

Latitude	0°	5°	10°	20°	30°	40°	50°	60°	70°	80°	90°
20 miles an hour .	∞	880	442	224	153	119	100	88	82	78	77
10 miles an hour .	∞	440	221	112	76	59	50	44	41	39	38
5 miles an hour .	∞	220	110	56	38	30	25	22	20	19	19

*W. M. Davis, "Elementary Meteorology."

"A body once set in motion under these conditions would continue moving forever, always changing its direction but never changing its velocity. If it were given a velocity of twenty miles an hour in any direction at Lat. 30°, it would describe a series of overlapping loops, gradually carrying it westward around the earth, but never passing outside of the parallels of 20° and 40°. If it were given a velocity of five or more miles an hour eastward at Lat. 5°, it would describe a scalloped path, oscillating back and forth across the equator, but never escaping beyond Lat. 5° in either hemisphere."

Some of this is correct, but most of it is not.

A body moving on a "globe" or "sphere" suffers no deflecting force and its path is not that described in the above quotation, nor is its relative velocity constant. A deflecting force occurs when a body is moving over a *spheroid* and is due to the contest between the gravitational and centrifugal forces. The path depends greatly upon the direction in which the body is launched. If launched with a velocity of twenty miles due east at 30° Lat., it would oscillate between 30° and 27° 43′, but would never go beyond 30°. If launched due west it would oscillate between 30° and 32° 9′ Lat. If launched due north or south it would oscillate between 31° and 29°.

We can derive the result, $\rho = \frac{v_r}{2\,\omega \sin \vartheta}$, from elementary considerations. If a body in latitude ϑ is moving over the earth in a direction AB, with a velocity v which would take it to B in the time dt if the earth were at rest, then since the earth is rotating under it in a counterclockwise direction in the northern hemisphere, instead of arriving at B, it will actually arrive at C. If the center of curvature of this path is at O, then, since AB is perpendicular to AO, it follows from elementary geometry that the angle CAB is half the angle COA. The arc $CA = v_r dt = \rho \angle COA$. $\angle CAB = \omega \sin \vartheta\, dt. = \frac{1}{2} \angle COA$. Hence, $\rho = \frac{v_r}{2 \sin \vartheta\, \omega}$.

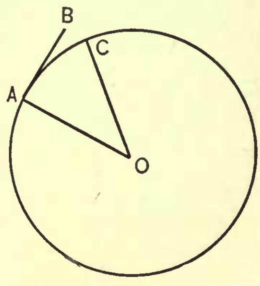

FIG. 15

However, this is merely an approximation, as we have pointed out, and does not hold if the relative velocity becomes excessive.

GENERAL CIRCULATION OF THE ATMOSPHERE

WITH a stationary world, hotter at the equator than at the poles, the circulation would be along the meridians. With a rotating world, the deflecting force, as we have seen, introduces east and west components, which result in the breaking up of what would otherwise be a simple circulation into practically independent fractions separated by parallels of latitude. In other words, instead of a single circulation there are several practically independent circulations. It is fortunate that this is so, for if the propelling forces which depend upon the differences of temperature were united along one line and summed together, such a blast would result, notwithstanding friction, that the world would be uninhabitable, especially in the higher latitudes.

EQUATORIAL CIRCULATION

As it is, we have to consider first the tropical circulation having a motive force which arises from differences of temperature no greater than those existing between the equator and about 30° Lat. This motive force which arises in the vertical components of the circulation is further reduced by the somewhat adiabatic expansions and compressions which it undergoes. The hot air arising from the equator loses some of its temperature by expansion and the return descending current becomes warmed by the descent, so that finally only a differential result is effective in driving the air currents. This driving force is about .006 of the weight of the air, while, as we have seen, the deflecting force is about $\frac{1}{4000}$ or .00025 of the weight. With these comparatively slight forces the circulation of the atmosphere is carried on as well as friction overcome.

The motive force of the equatorial circulations is thus the difference of temperature between the equator and some point to the north and south. Since in this zone the surface is approximately a cylinder, the circulation can take place in an approximately free manner. That is to say, the volumes of the circulating air at the two extremes will not be markedly changed, as would be the case if air were circulating between the hotter circumference and the colder center of a circle, which is the case in the polar circulation.

Given, therefore, a certain motive force between the equator and a point to the north, depending upon their difference of temperature, it is evident that the resultant velocity, which is proportional to the motive force, will develop deflective forces which will determine the path and the extent of the excursion to the north. If the extreme point to the north coincides with the temperature necessary to produce the driving force which will produce

the velocity necessary to reach it, there will be equilibrium. If at the point of recurving the temperature is insufficient to produce the necessary driving force to produce the requisite velocity, the upper border of the path will fall back until it finds a point capable of providing for the reduced velocity. If, on the other hand, the temperature potential is too great, the velocity will be increased and the upper border will progress upward until it finds a point where the driving force, velocity and upper border are in adjustment.

It will thus be seen that given a certain distribution of temperature on a planet, the width of the equatorial circulation is a function of this temperature distribution and its rotational velocity.

It so happens upon our earth that a distribution of temperature (average) ranging from 27° C. at the equator to 20° C. at the parallel of 30°, is just sufficient to drive (overcome all resistances to) the air, with a velocity sufficient to bring it up to the parallel of extreme temperature which is about 30° Lat.

The shape of this zone permits the air to take paths approximately like those of free bodies, though this is not exactly the case, since there is some constriction at the upper limit. Such a circulation will necessarily have a due east direction at the upper limit and a due west direction at the lower limit. At 10° Lat. a velocity of eighty miles an hour due west would carry a free body to about 25°, while a velocity of one hundred miles an hour would carry it to 30° before it recurved.

We know that velocities like these are constant in the upper layers over the equator. This fact, previously unsuspected, was revealed by the great eruption of Krakatoa, when a gale above the eight-mile level moving with a velocity of at least ninety miles an hour and probably more at the highest levels, carried the dust in a due west direction at least three times around the globe. Whymper noted during an eruption of Cotopaxi (1° 15′ S. Lat.) that the smoke rose vertically in still air until it attained a height of 40,000 ft. (eight miles), when it encountered a powerful current blowing due west, which carried it rapidly to the Pacific. In the higher levels, from 25° to 30°, the currents are constantly eastward, generally due east, with velocities of over a hundred miles. Thus the velocities and the upper limit of the equatorial circulation agree well with what we should expect from theory.

Meteorologists have called attention to the fact that the zone comprised between the two parallels of 30° contains exactly half of the atmosphere, and as the equatorial circulations are roughly measured by these limits, some connection between the two has been supposed to exist.

But as we have already pointed out, the width of this circulation depends upon the temperature distribution and the deflecting forces, the latter depending upon the rotational velocity of the earth.

The equatorial circulation of the planet Jupiter, if we may judge from the equatorial bands, is much less than half, which is probably due to its

great rotational velocity, a complete revolution being performed in ten hours.

We shall now attempt to build up the equatorial circulation from theory, and from observation as far as this is possible from data at hand. We shall suppose that the equatorial stream is a broad band extending 10° north and south of the equator. From the polar borders of this stream currents start out at a high level with the velocity of the stream and practically due west. After making the circuit of the equatorial circulation these currents return into the main stream again in a due west direction, but at a

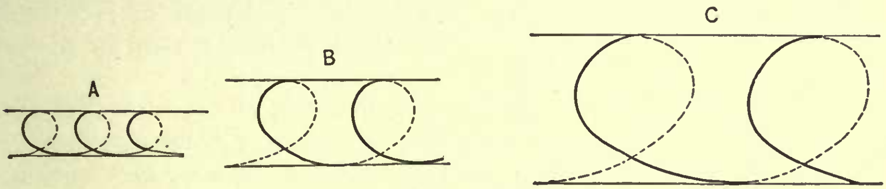

FIG. 16

lower level. Since the outgoing and incoming masses of air must be equal, the level separating them must be at about the level of half atmosphere, something like three miles.

We shall suppose at the level of half atmosphere, therefore, a small flat tube *A*, Fig. 16, where the full lines represent the upper outgoing currents and the dotted lines the lower returning current. The velocities here are low and the polar extension, therefore, small. Over this small tube we shall slide the larger flat tube *B*. The outgoing currents here start from a higher level and the incoming currents reach the main stream at a lower level. The

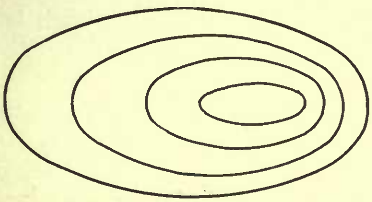

FIG. 17

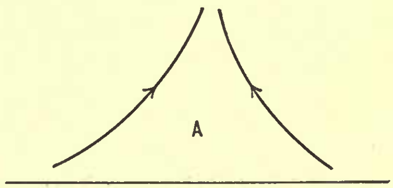

FIG. 18

velocities are also higher and the extension north correspondingly greater. Over this tube we slide the larger tube *C* and so on until the highest levels are reached for the outgoing currents, the lowest levels for the incoming currents, and the maximum velocity with maximum excursion towards the pole.

The vertical section of these tubes would be something like Fig. 17. The tubes would be crowded together on the equatorial border, but separated more and more on the polar border. This allows the volumes to accommodate themselves to the moderate constriction towards the poles.

At the equator, on the surface of the earth, and extending to each side for about 10° is a wedge-shaped space of calms, the borders of this space *A* being limited by the incoming and ascending lower currents. The height of this space at the middle is about eight miles. (Fig. 18.)

The equatorial circulation would, therefore, be represented by a series of flat tubes, crowded together on their equatorial border, but spreading out on their polar borders. Each tube is a practically independent circulation where the currents spiral around in a clockwise direction gradually moving to the west. The core would carry the lowest velocities, while the velocities increase from the core outward. This is necessitated from the fact that a current must be continuous, not being capable of being crossed by another current.

Such an arrangement satisfies the requirements of the vertical circulation, which is driven by the differences of temperature at the extremes. It satisfies the conditions of continuity and the stream lines are such as would be produced by the deflecting forces due to the earth's rotation.

We must next inquire how such a theoretical structure agrees with observation. We shall call the incoming lower currents the trades; the outgoing upper currents the antitrades.

Just beyond 10° Lat. we should find the trade decreasing with the height until we struck the beginnings of the lowest and slowest antitrades going to the northwest. This has been observed from the motion of clouds. At a little higher latitude we should, after passing through the trades, meet an antitrade blowing first to the east and, on rising higher, one blowing to the northeast and finally one blowing to the north. This has not been observed because no sounding observations to great altitudes have been conducted in low latitudes. At present we do not know whether such is the fact or not, but for the benefit of our theory we may say that we do not know that such is *not* the fact.

We have said that the first antitrades met with, on ascending, should be to the east. This is confirmed by observations in low latitudes. The smoke of volcanoes, after breaking through the trades, has repeatedly been seen to go to the eastward. The smoke of St. Vincent was in May, 1812, carried eastward to Barbados. The smoke of Consequina in Nicaragua, in 1835, went E. N. E. to Jamaica. Farther north, at the summits of Teneriffe and Mauna Loa a strong S. W. antitrade is always observed. Humboldt states that at the top of Teneriffe he could hardly stand before this southwest blast.

In Java and Sumbawa smoke from volcanoes has drifted east. We have thus direct evidence of currents to the N. W., N. E., and E. as required by our theory. No direct observation of a current due north in the antitrades seems to be at hand, although of necessity such a current must exist. The need of sounding observations for the lower latitudes, such as

have been conducted at Blue Hill by Rotch and Clayton, and in Sweden is thus apparent. The upper atmosphere is in great part unexplored. To sum up the equatorial circulation seems to be represented by a curious system of flat tubes, the outer ones containing all the inner ones, yet not concentrically. The stream lines in each tube are a system of spirals which progress to the westward and have for their horizontal projections the series of overlapping loops we have represented in the figure. The vertical projections are a series of flat ovals, lying completely within each other and crowded together at the equatorial end. The conditions fulfilled by such a circulation are the following:

1. The extra weight of the polar end is continually forcing a current along the lower level to the equatorial end, where it rises and continually slides back again to the polar end along its upper level.

2. The path is shaped by the deflective forces, which here have nearly free play.

3. There is at no point any discontinuity. That is, we can go from one point to another by insensible gradations, both as to velocity and direction of current.

4. Observations, so far as they are at hand, support the theory. There are at present no contradictions.

POLAR CIRCULATION

We shall next consider the polar circulation. We have here to consider a distribution of temperature practically on a plane, a circle, which changes gradually from the circumference, where it is a maximum, to the center, where it is a minimum.

The general circulation here must be of the nature of a sinking in at the center, an outflow along the lower levels to the circumference, where it rises and gradually makes its way back to the pole over the upper levels. The current directly downward near the center must be considerable, although, owing to the spreading out of the current, there would be a cone-shaped space of calm at the pole itself. As the stream flows off from the pole it would acquire a westward component relatively to the earth, which would develop a deflective force to the right, tending to bring it back to the pole with a short turn. But the deflective force here does not have free play. It is opposed and overcome at all points by the outstreaming air from the pole. Consequently, the air must be forced out ever more and more, always with a westward component, until it reaches a point where the deflective force is at last equal to the outward pressure gradient. At this point it circulates directly to the westward, around the pole in momentary equilibrium. But it now rises, becoming warmer, and as at a higher level the outward gradient is weakened (there is not so much outflowing air here)

the stream gradually falls back towards the pole. If the deflective forces had free play here, the paths would be nearly circular and of short diameter, but the inward motion is checked by the air already in advance seeking its way to the pole, so that it turns towards the pole very gradually instead of sharply, as would otherwise be the case. The velocities are no doubt considerable at the upper levels, but in general they are kept in check by friction on the outgoing half and by the back pressure of the returning half. The important point is that the deflective forces, although very pronounced, are held in check by the choking forces due to the constriction of volume towards the center. They are, therefore, unable to shape the paths of the streams, as they do near the equator. The circulation, therefore, consists of a spiraling outward until the circle of equilibrium is reached and then a spiraling inward, but always in the same direction, viz., to the westward. Of course, the pressure is greatest at the center and least at the circumference.

The work done by a heat engine is proportional to the difference between the two extreme temperatures at which it works. Consequently, the friction overcome by the polar whirl in order to maintain its energy of rotation constant is proportional to the difference between the temperature of the pole and that of the circle of equilibrium.

This ideal polar circulation is realized only approximately in the case of the earth. The unequal distribution of land and water, especially at the north pole, is a greatly disturbing factor. But even if the surface of the earth were homogeneous at the poles, owing to the inclination of its axis and the change of seasons, the pole would rarely be the place of minimum temperature, and the position of the circle (if circle it were) of equilibrium would be extremely variable. Thus, under the actual conditions, the determination of the polar circulation defies all analysis.

Still there is some agreement between the theoretical case we have considered and what is actually found. On the whole, the surface winds come out of the pole and the components to the west are very marked. The particular distribution of temperatures at the earth's poles combined with its velocity of rotation results in the circle of equilibrium lying on an average about 25° from the poles.

From the homogeneity of its surface, the south pole would naturally make a nearer approach to theory than the north pole. We quote the following from Ward ("Climate"):

"The rapid southward decrease of pressure, which is so marked a feature of the higher latitudes of the southern hemisphere on the isobaric charts of the world, does not continue all the way to the south pole. The steep poleward pressure gradients of these southern oceans end in a trough of low pressure, girdling the earth at about the antarctic circle. From here the pressure increases again towards the south pole, where a permanent

inner polar anticyclonic area is found, with outflowing winds deflected by the earth's rotation into easterly and southeasterly directions. A chart of the south polar isobars for February (after Sir John Murray and Dr. Buchan) published in 1898, showed a pressure of 29.00 inches in the low pressure girdle, and the isobar 29.50 inches around the inner polar area. These easterly winds have been observed by recent expeditions which have penetrated far enough south to cross the low pressure trough. The limits between the prevailing westerlies and the outflowing winds from the poles, easterlies, vary with the longitude and migrate with the seasons. The change in passing from one wind system to the other is easily observed.

"The Belgica, for example, in latitudes 69½°-71½° S. and longitudes 81°-95° W., was carried towards the west by the easterly winds of summer, and in winter was driven east by the westerlies, and then again to the west. The Belgica thus lay in winter on the equatorial, and in summer on the polar side of the trough of low pressure. The seasonal change in wind direction was very marked, being almost monsoon-like in character. On the other hand, the English expedition at 77° 50′ S. was persistently on the polar side of the trough, with dominant S., S. E., and E. winds. The smoke from Mt. Erebus, however, showed prevailing southwesterly currents.* The German expedition on the Gauss was also under the régime of easterly winds during its stay in winter quarters. The Belgica had fewer calms than some stations nearer the pole. The south polar anticyclone with its surrounding low pressure girdle migrates with the season, the center apparently shifting poleward in summer and towards the eastern hemisphere in winter. The cloudier winds are poleward; the clearer winds blow out from the pole. The out-flowing winds from the polar anticyclone sweep down across the inland ice and are usually cold."

Drygalski found foehn-like winds in the antarctic regions, blowing a gale mostly from S. and E. It will be seen that all this is practically a description of such a circulation as we have deduced from theory.

MIDDLE CIRCULATION

We shall lastly consider the nature of the circulation between the polar circle of equilibrium and the polar border of the equatorial circulation. We have already recognized two distinct circulations, viz., the equatorial circulation and the polar circulation. The former extends over something like 30°, while the latter extends over something like 25°. We shall now examine the intervening zone.

At the polar border of the equatorial circulation the currents are curving more sharply under the influence of the deflecting force than at any other point. They are here going directly east. The neighboring air on the outside is taken along with them by friction in the same direction, and, there-

* i.e., Moving towards the S.W.

fore, tends under the influence of the deflecting force to be carried towards the equator. But the equatorial circulation is full, incapable of receiving any more air. These outer currents, therefore, exert a pressure towards the equator without being able to move in that direction.

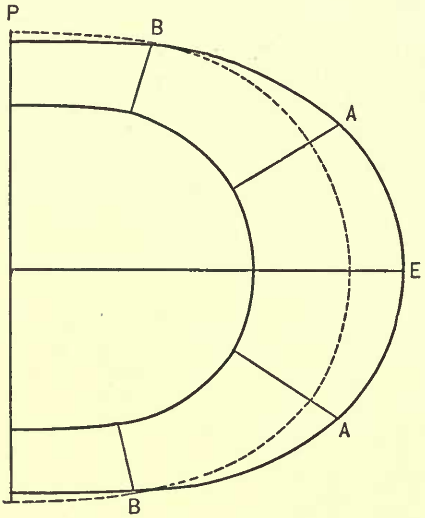

FIG. 19

There must, therefore, be along this border a ridge of high pressure. In Fig. 19 the dotted line is supposed to represent the upper surface of the atmosphere with a uniform temperature throughout of 0° C. The full line represents this surface under actual conditions. The proportions are greatly exaggerated.

The surface is highest at the equator, whence it slopes down gradually to the poles where the height is least. At the polar border of the equatorial circulation, *A,* the pressure is greater than at any point on the earth's surface. At the pole is another place of maximum pressure, though this maximum is not as great as that at *A*. At *B*, the border of the polar circulation is a place of minimum pressure, as is also the equator. At the border of the south polar circulation, the pressure is less than at any point on the earth's surface. The minimum at the border of the north polar circulation is about the same as that at the equator, perhaps a little greater. From *A* to *B,* the pressure falls considerably. From *B* to *P* the pressure rises.

Notwithstanding the temperature gradient from *B* to *A,* the pressure at *A* is much greater than at *B* because, although volume for volume the air at *A* is considerably lighter than that at *B,* still there is much more of it and it rises to a greater height. The path, therefore, of a particle in the middle circulation is wholly independent of the temperature gradient, but depends upon the gravitational gradient or slope down which it tends to fall to the pole. In other words, if it were not for the centrifugal force, the warmer column of air at *A* would prevail against the colder column at *B*. Actually they are balanced in a rather delicate equilibrium.

Let us follow the course of a particle just outside the equatorial circulation. Since it cannot cross the high pressure ridge, which is heaped full, it will continue along parallel to this ridge as long as the centrifugal force is equal to the component of gravity urging it down hill. The force tending to reduce its centrifugal force is friction, and this will be more pronounced in the lower levels than at the upper. Close to the ground we know that

the velocity is very much reduced. This reduction of velocity from friction will cause it to move towards the pole especially in the lower levels. Since the moment of momentum is preserved it will soon acquire a velocity which will hold it momentarily in equilibrium in its new position. The air from above sinks in order to take the place of the air which has moved towards the pole. This relieves the gravitational gradient above and the air in the upper levels which has been pressing towards the equator takes the place of the air which has sunk. A circulation is thus started which consists in a gradual movement towards the pole in the lower levels and a movement outward in the higher levels. The air moving poleward along the lower levels displaces the air in front of it by raising it. In its elevated position it is less subject to friction and quickly assumes the higher velocities of the upper layers. It is now ready to move outward. The movement is not due to its own temperature, but against or in spite of its own temperature gradient. The driving force is derived from the heat or temperature gradient of the equatorial circulation. The increase of velocity of a particle in the higher levels at a high latitude as it moves along this level to a lower latitude is effected by the friction of the equatorial circulation on the equatorial border of the middle circulation.

It is not to be supposed that there is a definite circulation here by which a particle sinks near the lower border and, after moving along the surface to the extreme upper border, rises there to make its way along the higher levels to the lower border again. There is not a single simple circulation, but rather a series of circulations all moving to the east, only to a slight extent wholly independent of each other, and all more or less merged and changing from time to time. On the discs of Jupiter and Saturn we can detect by the aid of a telescope several such semi-independent circulations forming the middle circulation, and all undoubtedly move to the east.

The results of such a circulation are that the highest pressure is at the lower border, and the lowest pressure at the upper border. The surface winds will in general be somewhat south of west, at least in the lower latitudes, while the upper currents will in general be somewhat north of west.

With the shifting of the circulations with the seasons slight changes may be produced in the direction of the currents in the lower levels, so that there may be at times W. and N. W. winds, but the general trend is not far from west. Any one living in temperate latitudes with the slightest powers of observation must have noticed the general eastward direction of all winds.

Professor James Thomson pointed out in a general way that the prevalence of S. W. winds in the north temperate zone might be explained by the poleward pressure at the surface combined with the differential velocity of the earth.*

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* British Association Report, 1857.

Lord Kelvin has also called attention to the analogy existing between this circulation and a cylindrical vessel containing water which is set in rotation by stirring. The centrifugal force heaps up the water around the edges. The friction on the bottom retarding the rotatory velocity and thus decreasing the centrifugal force of the particles on the bottom renders them unable to withstand the increased pressure at the edges due to the heaping up of the water there. They are thus forced towards the center, where on rising to the surface they quickly assume the velocities of their neighbors and move outwards under the influence of centrifugal force, which is here less hindered by friction. There is thus a continuous circulation, consisting of a spiraling outward along the upper layers, a sinking at the edge and then a spiraling inward along the lower layers.*

Ferrel criticises this explanation as follows, in his "Motion of Fluids and Solids Relative to the Earth":

"At the meeting of the British Association for the Advancement of Science, in the year 1857, Mr. Thomson read a short paper in which he explains this accumulation of the atmosphere and the consequent reversion (i.e., northing) of the lower strata of the atmosphere in middle latitudes, as arising from the centrifugal force of the eastward motion of the atmosphere, and illustrates the effect of such a force by means of a gyrating vessel of water in which the surface water recedes from the center, while at the bottom there is a flowing towards it. In this paper nothing is said of the influences of the earth's rotation, and if he means that the effect is produced simply by the centrifugal force arising from the eastward motion of the atmosphere relative to the earth's surface, independent of the earth's rotation, the force would not be great enough to produce any sensible effect. For examining the expression of the force which produces this effect, in Section 100, it is seen that it depends upon the earth's rotation, since $\frac{d\varphi}{dt}$, the angular velocity of the atmosphere relative to the earth, is small in comparison with $2u$, which is double the velocity of the earth's rotation."

This criticism of Ferrel's is just, for the centrifugal force which causes the heaping up at the border between the equatorial and middle circulations is the function $R \sin \vartheta \cos \vartheta (\omega^2 - \dot{\psi}^2)$ and not, as Kelvin supposed, $R \sin \vartheta \cos \vartheta \dot{\psi}_r^2$, where $\dot{\psi}_r$ is the angular velocity of the circulation relatively to the earth.

With this correction, however, there can be no doubt but that Kelvin's explanation of the middle circulation is the correct one.

To recapitulate, then, the middle circulation derives its velocity relatively to the earth from the equatorial circulation by friction along the border. This rotatory velocity causes a heaping up of the air at the equatorial border

* Lord Kelvin (then Professor William Thomson). Report of the British Association for the Advancement of Science, 1857.

with increased pressure, and this pressure forces the air along the surface towards the pole. The rotatory velocity of the lower layers being diminished by friction, this poleward motion is less opposed by centrifugal force. On arriving at their polar border the particles rise and quickly acquire the increased velocity of the upper layers. Centrifugal force now becomes predominant and drives them outward along the upper levels, and thus the circulation is established.

We have thus a satisfactory theory of the general circulation of the atmosphere which agrees well with observed facts. Some of the deductions have not yet been verified because our knowledge of the motions of the upper atmosphere is so incomplete. The knowledge we have derived from the smoke of volcanoes and sounding observations agrees with what we should expect.

PLANETARY CIRCULATIONS

The proposition would seem to be established that an atmosphere on a rotating planet, having its maximum surface temperature at the equator and its minimum surface temperature at the pole, naturally divides itself into six independent circulations. The most important are the two equatorial circulations which, by its temperature potential, carries on its own as well as the middle circulation. The middle circulations may according to circumstances divide themselves up into several more or less differentiated circulations, though the borders between these circulations are neither permanent nor marked.

The polar circulations are driven by their own temperature potentials, but differ from the equatorial circulations in that while in the latter deflective forces have free play, in the former they are overcome by gravitational forces. In the middle circulations the temperature potential is overcome by centrifugal and gravitational forces.

The width of the equatorial and polar circulations depends upon their temperature potentials and the rotatory velocity of the planet.

When we have spoken of independent circulations, it is not to be understood that this is absolutely the case. There is, of course, always more or less of interchange of particles between the different circulations, and, given sufficient time, every particle will at some time or other occupy every possible position. Notwithstanding, we can consider the circulations as practically independent—a very fortunate circumstance, as by dividing up the circulations the earth is rendered habitable, which would not be the case with a single circulation.

It has been supposed that because the weight of the atmosphere was less at the equator than at the tropics, there must be a hollowing out or actual deficiency at the upper surface there. On the contrary, there is an actual bulging at the equator at all levels, since it is down the slope of this bulge

that the antitrades slide to the tropics. A general principle of some importance regarding the atmosphere as a whole is the following. If starting with the atmosphere at a uniform temperature and giving the frictional forces time enough to bring all particles to a state of relative rest with the earth, we gradually brought the surface of the earth to its actual temperatures, then the circulations which would be set up would be wholly the result of temperature potentials. These motive forces acting on the whole along meridians, the moments of those forces about the axis of the earth would always be zero. Consequently the velocity of the atmosphere as a whole about the axis of the earth could not be increased and the sum of the moments of the velocities of all the particles would always be equal to the sum of the moments of the velocities of the earth's surface into the mass of the atmosphere.

Ferrel expresses this in the following manner:

"It is evident, however, that the east and west motions* of the atmosphere at the earth's surface multiplied into its distance from the axis of rotation, must be zero; else the velocity of the earth's rotation would be continually accelerated or retarded, which cannot arise from any mutual action between the surface of the earth and the surrounding atmosphere."

ANNUAL MEAN OF PRESSURES FOR LATITUDE

Lat.	P. Mms.	Lat.	P. Mms.	Lat.	P. Mms.	Lat.	P. Mms.
+80°	760.5	40°	762.0	0°	758.0	40°	760.5
75	760.0	35	762.4	−5	758.3	45	757.3
70	758.6	30	761.7	10	759.1	50	753.2
65	758.2	25	760.4	15	760.2	55	748.2
60	758.7	20	759.2	20	761.7	60	743.4
55	759.7	15	758.3	25	763.2	65	739.7
50	760.7	10	757.9	30	763.5	70	738.0
45	761.5	5	758.0	35	762.4		

* Meaning velocities relatively to the earth.

CYCLONES

We shall next take up the subject of abnormal disturbances of the atmosphere, such as cyclones, tornadoes, thunderstorms, etc. Besides their scientific interest, these phenomena possess for us an intensely human interest. For this reason a digression of a purely descriptive character may be permitted before we take up their formal mathematics, especially since comparatively few persons, including meteorologists, have ever witnessed or ever will witness a tropical hurricane.

The following description of a West Indian hurricane by Alexander Hamilton is interesting from its accuracy and its quaintness. It was the first writing of this distinguished man ever published. He was at the time quite a young man, visiting friends in St. Croix, D. W. I. On August 31, 1772, a hurricane of small diameter but considerable violence passed over St. Croix, and the following description, which Hamilton sent to his father, was published in the local paper. It shows the effect produced upon a bright young mind by one of these terrific disturbances.

"St. Croix, September 6, 1772.

"Honoured Sir: I take up my pen just to give you an imperfect account of the most dreadful hurricane that memory or any records whatever can trace, which happened here on the 31st ultimo at night.

"It began about dusk, at north, and raged very violently till ten o'clock. Then ensued a sudden and unexpected interval, which lasted about an hour.* Meanwhile, the wind was shifting round to the southwest point, from whence it returned with redoubled fury and continued so till near three o'clock in the morning. Good God! What horror and destruction—it's impossible for me to describe, or you to form any idea of it. It seemed as if a total dissolution of nature was taking place. The roaring of the sea and wind—fiery meteors falling about in the air—the prodigious glare of almost perpetual lightning—the crash of the falling houses, and the ear-piercing shrieks of the distressed were sufficient to strike astonishment into angels. A great part of the buildings throughout the island are leveled to the ground—almost all the rest very much shattered—several persons killed and numbers utterly ruined—whole families running about the streets unknowing where to find a place of shelter—the sick exposed to the keenness of water and air—without a bed to lie upon—or a dry covering to their bodies—our harbour is entirely bare. In a word, misery in all its most hideous shapes

* The cyclone center passed directly over St. Croix.

spread over the whole face of the country. A strong smell of gunpowder added somewhat to the terrors of the night; and it was observed that the rain was surprisingly salt. Indeed, the water is so brackish and full of sulphur that there is hardly any drinking it. My reflections and feelings on this frightful and melancholy occasion are set forth in the following self-discourse.

"Where now, oh! vile worm, is all thy boasted fortitude and resolution? What is become of thy arrogance and self-sufficiency? Why dost thou tremble and stand aghast? How humble, how helpless, how contemptible you now appear. And for why? The jarring of the elements—the discord of the clouds? Oh, impotent presumptuous fool! How darest thou offend that Omnipotence whose nod alone were sufficient to quell the destruction that hovers over thee, or crush thee into atoms? . . . Let the earth rend, let the planets forsake their course, let the sun be extinguished and the heavens burst asunder—yet what have I to dread? My staff can never be broken—in Omnipotence I trust. . . . Hark! Ruin and confusion on every side. 'Tis thy turn next: but one short moment—even now—Oh! Lord, help—Jesus, be merciful! Thus did I reflect and thus at every gust of the wind did I conclude—till it pleased the Almighty to allay it. . . .

"I am afraid, sir, you will think this description more the effect of imagination than a true picture of realities. But I can affirm with the greatest truth that there is not a single circumstance touched upon which I have not absolutely been an eye-witness to."

As another example of the fearful human interest attaching to a tropical cyclone, we give a description of the hurricane of August 10, 1856, which devastated Last Island on the Louisiana coast. Forty-four years later (September 6, 1900) a similar cyclone passed some miles farther to the west and devastated Galveston with a much greater loss of life. The description is by that incomparable word-painter Lafcadio Hearn, who must have witnessed such a cyclone to have described it so accurately.

"Then one great noon, when the blue abyss of day seemed to yawn over the world more deeply than ever before, a sudden change touched the quicksilver smoothness of the waters—the swaying shadow of a vast motion. First the whole sea-circle appeared to rise up bodily at the sky; the horizon-curve lifted to a straight line; the line darkened and approached—a monstrous wrinkle, an immeasureable fold of green water, moving swift as a cloud shadow pursued by sunlight.

"But it had looked formidable only by startling contrast with the previous placidity of the open: it was scarcely two feet high; it curled slowly as it neared the beach, and combed itself out in sheets of woolly foam with a low, rich roll of whispered thunder.*

"Swift in pursuit another followed—a third—a feebler fourth: then the

* The advance waves running before the cyclone.

sea only swayed a little and stilled again. Minutes passed and the immeasurable heaving recommenced—one, two, three, four—seven long swells this time;—and the Gulf smoothed itself once more. Irregularly the phenonemon continued to repeat itself, each time with heavier billowing and briefer intervals of quiet—until at last the whole sea grew restless and shifted color and flickered green; the swells became shorter and changed form. Then from horizon to shore ran one uninterrupted heaving—one vast green swarming of snaky shapes, rolling in to hiss and flatten upon the sand. Yet no single cirrus-speck revealed itself through all the violet heights: there was no wind!—you could have fancied the sea had been upheaved from beneath.

"Still the sea swelled and a splendid surf made the evening bath delightful. Then, just at sundown, a beautiful cloudlike bridge grew up and arched the sky with a single span of cottony pink vapor, that changed and deepened color with the dying of the iridescent day. And the cloud bridge approached, stretched, strained and swung round at last to make way for the coming of the gale—even as the light bridges that traverse the dreamy Têche swing open when the luggermen sound through their conch-shells the long, bellowing signal of approach.

"Then the wind began to blow; it blew from the northeast, clear, cool. It blew in enormous sighs, dying away at regular intervals, as if pausing to draw breath. All night it blew: and in each pause could be heard the answering moan of the rising surf—as if the rhythm of the sea moulded itself after the rhythm of the air—as if the waving of the water responded precisely to the waving of the wind—a billow for every puff, a surge for every sigh.

"The waves were running now at a sharp angle to the shore; they began to carry fleeces, an immeasurable flock of vague green shapes, wind-driven to be despoiled of their ghostly wool. Far as the eye could follow the line of the beach, all the slope was white with the great shearing of them. Clouds came, flew as in a panic against the face of the sun and passed. All that day and through the night and into the morning again the breeze continued from the northeast, blowing like an equinoctial gale.

"Colossal breakers were herding in, like moving leviathan-backs, twice the height of a man. Still the gale grew, and the billowing waxed mightier, and faster and faster overhead flew the tatters of the torn cloud. The gray morning of the 9th wanly lighted a surf that appalled the best swimmers; the sea was one wild agony of foam, the gale was rending off the heads of the waves and veiling the horizon with a fog of salt spray. Shadowless and gray the day remained; there were mad bursts of lashing rain. Evening brought with it a sinister apparition, looming through a cloud rent in the west—a scarlet sun in a green sky. His sanguine disc, enormously magnified, seemed barred like the body of a belted planet. A moment and the crimson specter vanished and the moonless night came.

"Then the wind grew weird. It ceased being a breath; it became a voice moaning across the world—hooting—uttering nightmare sounds—Whoo! Whoo! Whoo!—and with each stupendous owl-cry the moving of the waters seemed to deepen, more and more abysmally, through all the hours of darkness. From the northwest the breakers of the bay began to roll high over the sandy slope into the salines; the village bayou broadened to a bellowing flood. So the tumult swelled and the turmoil heightened until morning—a morning of gray gloom and whistling rain. Rain of bursting clouds and rain of wind-blown brine from the great spuming agony of the sea.

"Cottages began to rock. Some slid away from the solid props upon which they stood. A chimney tumbled. Shutters were wrenched off; verandas demolished. Light roofs lifted, dropped again and flapped into ruin. Trees bent their heads to earth. And still the storm grew louder and blacker with every passing hour.

"Almost every evening throughout the season there had been dancing in the great hall: there was dancing that night also. The population of the hotel had been augmented by the advent of families from other parts of the island, who found their summer cottages insecure places of shelter; there were nearly four hundred guests assembled. Perhaps it was for this reason that the entertainment had been prepared upon a grander plan than usual, that it assumed the form of a fashionable ball. And all those pleasure-seekers—representing the wealth and beauty of the Creole parishes—whether from Ascension or Assumption, St. Mary's or St. Landry's, Iberville or Terrebonne, whether inhabitants of the multi-colored and many-balconied Creole quarter of the quaint metropolis, or dwelling in the dreamy paradises of the Têche—mingled joyously, knowing each other, feeling in some sort akin—whether affiliated by blood, connaturalized by caste or simply inter-associated by traditional sympathies of class sentiment and class interest.

"Perhaps in the more than ordinary merriment of that evening something of nervous exaltation might have been discovered—something like a feverish resolve to oppose apprehension with gayety, to combat uneasiness by diversion. But the hours passed in mirthfulness; the first general feeling of depression began to weigh less and less upon the guests; they had found reason to confide in the solidity of the massive building; there were no positive terrors, no outspoken fears; and the new conviction of all had found expression in the words of the host himself—'Il n'y a rien de mieux à faire que s'amuser.' Better to seek solace in choregraphic harmonies, in the rhythm of gracious motion and perfect melody, than hearken to the discords of the wild orchestra of storms, wiser to admire the grace of Parisian toilets, the eddying of trailing robes with its fairy-foam of lace, the ivorine loveliness of glossy shoulders and jeweled throats, the glimmering of satin-

slippered feet—than to watch the raging of the flood without, or the flying of the wrack.

"Night wore on; still the shining floor palpitated to the feet of the dancers; still the pianoforte pealed, and still the violins sang—and the sound of their singing shrilled through the darkness, in gasps of the gale.

" 'Waltzing!' cried a sea captain, 'God help them! God help us all now! The wind waltzes to-night with the sea for his partner.'

"O the stupendous Valse-Tourbillon! O the mighty dancer! One-two-three! From the northeast to east, from east to southeast, from southeast to south; then from the south he came, whirling the sea in his arms. Some one shrieked in the midst of the revels; some girl who had found her pretty slippers wet. What could it be? Then streams of water were spreading over the level planking, curling about the feet of the dancers.

"What could it be? All the land had began to quake, even as but a moment before the polished floor was trembling to the pressure of circling steps; all the building shook now; every beam uttered its groan. What could it be? There was a clamor, a panic, a rush to the windy night. Infinite darkness above and beyond; but the lantern beams danced far out over an unbroken circle of heaving and swirling black water. Stealthily, swiftly, the measureless sea-flood was rising.*

"For a moment there was a ghastly hush of voices. And through that hush there burst upon the ears of all a fearful and unfamiliar sound, with volleying lightnings. Vastly and swiftly, nearer and nearer it came—a ponderous and unbroken thunder-roll, terrible as the long muttering of an earthquake.

"The nearest mainland—across mad Caillon Bay to the sea marshes—lay twelve miles north; west, by the gulf, the nearest solid ground was twenty miles distant. There were boats, yes! but the stoutest swimmer might never reach them now!

"Then rose a frightful cry—the hoarse, hideous, indescribable cry of hopeless fear—the despairing animal-cry man utters when suddenly brought face to face with Nothingness, without preparation, without consolation, without possibility of respite—*Sauve qui peut.*

"And then—then came, thundering through the blackness, the giant swells, boom on boom! One crash! The huge frame building rocks like a cradle, seesaws, crackles. Another! Chandeliers splinter; lights are dashed out; a sweeping cataract hurls in; the immense hall rises—oscillates—twirls as upon a pivot—crepitates—crumbles into ruin. Crash again! the swirling wreck dissolves into the wallowing of another monster billow; and a hundred cottages overturn, spin in eddies, quiver, disjoint and melt into the seething.

* The cyclone wave carried along by the center of the cyclone.

"So the hurricane passed, tearing off the heads of prodigious waves, to hurl them a hundred feet in air—heaping up the ocean against the land—upturning the woods. Bays and passes were swollen to abysses; rivers regorged; the sea marshes were changed to raging wastes of water. Before New Orleans the flood of the mile-broad Mississippi rose six feet above high water mark. One hundred and ten miles away Donaldsonville trembled at the towering tide of the Lafourche. Lakes strove to burst their boundaries. Far off river steamers tugged wildly at their cables—shivering like tethered creatures that hear by night the approaching howl of destroyers. Smokestacks were hurled overboard, pilot houses torn away, cabins blown to fragments. And over roaring Kaimbuck Pass—over the agony of Caillou Bay—the billowing tide rushed unresisted from the gulf—tearing and swallowing the land in its course—ploughing out deep-sea channels where herds had been grazing but a few hours before—rending islands in twain—and ever bearing with it, through the night, enormous vortex of wreck and wan drift of corpses."

A whirl of air, in the same direction as the earth turns at the point where the whirl exists, is called a cyclone. A whirl in the opposite direction is called an anticyclone. We have seen that the tendency of a moving mass on the earth's surface is to turn towards the right in the northern hemisphere, and to the left in the southern hemisphere. Projected upon the plane of the equator these directions are the opposite of the earth's rotation or anti-cyclonic.

Before we come to consider the formation of a cyclone, it may be stated that in such whirls the pressure is always considerably less in the center than around the edges. Consequently there is always a force urging the gyrating stream towards the center. Motion directly towards the center is prevented by the deflecting force which turns it to the right in the northern hemisphere. Hence in either hemisphere a local diminution of pressure must result in a cyclonic whirl. The deflective force we have seen to be approximately $2v\,\omega \sin \vartheta$, where v is the relative velocity and the other symbols have their usual significance. The deflective forces and the pressure gradients are thus opposed to each other. If the component of the deflective force away from the center is equal to the pressure gradient towards the centre, then a particle will be urged neither inward nor outward and will tend to gyrate in a circle about the center of low pressure. If the pressure gradient is predominant then the particle will spiral inward: if the deflective force is predominant, then the particle will spiral outward. In the dynamics of the middle circulation of the atmosphere, we have had an example of such constrained motion, where the deflective force continually urges the stream outward towards the equator, while the pressure gradient opposes this motion. From this point of view the middle circulation is as truly a cyclone as the smaller and more violent whirls which occur at different points of the

earth's surface. The middle circulation is, therefore, cyclonic, while the polar circulation is anticyclonic, and the equatorial circulation is neither.

There are two principal theories as to the formation of cyclonic whirls. The first postulates a simple local heating of some extent which gives rise to a convectional ascending current. As the bottom of an ascending current is always a locus of diminished pressure for reasons already given, we have here our center of low pressure and the rest follows easily. The inward pressure is at first predominant and forces the currents to spiral inward towards the center, which they turn with a high velocity, and rising to levels where the inward pressure gradient becomes weakened, they finally spiral outward at the top. The whirl is cyclonic throughout and not, as has been frequently stated, anticyclonic at the top. This theory is rather plausible and it is possible that some cyclones are formed in this way. On the whole, however, it is far from satisfactory.

Ferrel, who was one of its advocates, says: "On account of the non-homogeneity of the earth's surface, comprising hills and valleys, land and water, and dry and marshy areas, all with different radiating and absorbing powers, and also on account of the frequent irregular and varying distribution of clouds, it must often happen that there are considerable local departures of temperature from that of the surrounding parts; and if it should so happen, as it frequently must, that this area is of a somewhat circular form, and the air has a temperature higher than that of the surrounding part of the atmosphere, then we have the conditions required to give rise to a vertical circulation, with an ascending current in the interior, as described above." This is eloquent pleading for a rather weak case. Few, if any, of the severer cyclones (the tropical ones) originate on land, where non-homogeneity and the greatest local differences of temperature are found. Tropical cyclones usually originate on the sea, just within the polar borders of the doldrums, where no great local differences of temperature exist.

While, undoubtedly, heat is necessary for the continued existence of a cyclone, for a cyclone is after all a heat engine, yet in the first instance it would seem that they are usually started by dynamical agencies other than heat. A whirl of leaves on a gusty day does not depend upon any local difference of temperature at the point of origin, but is due to two or more currents meeting obliquely.

We thus come to the second theory, the dynamical one, which refers the origin of cyclones to two opposing sheets of wind meeting along an oblique line. If we spread some light powder on a table and blow on it simultaneously with two bellows from opposite directions, but a little oblique to each other, the powder will rise from the table in a whirl, and with a little care we can determine the direction of the whirl.

Now the doldrums shift their position with the seasons, moving north and south. In the Atlantic the whole belt of doldrums, although of varying

width, always remains to the north of the equator, while in the Pacific and Indian oceans it is alternately wholly to the north or south of the equator. The equatorial border of the trades which limit this belt, owing to the earth's deflective force, blow N. W. and S. W. when on the wrong side of the equator. That is, a northeasterly trade becomes deflected into a northwesterly trade on crossing the equator, while the southeasterly trade turns into a southwesterly one when north of the equator.

While performing this seasonal shift north and south, it frequently happens that the rearward trade overtakes the other one, which has not retreated fast enough to keep out of its way. It thus happens that two sheets become obliquely opposed to each other and a whirl is thus set up. Severe tropical cyclones originate almost without exception while this shift is being effected. Thus the tropical cyclones of the northern hemisphere begin about July and continue up to December, while the belt of doldrums is making its southern retreat. The equatorial circulation is, during this time, becoming accelerated owing to an increasing temperature gradient between its limits. The oncoming N. E. trade is, therefore, more vigorous than the retreating S. W. trade and frequently overtakes it. The result is a cyclone or a series of cyclones along the opposing borders. Since the doldrums never cross the equator in the Atlantic, always remaining to the north of the equator, there are no cyclones in the South Atlantic within the tropic zone.

The same thing takes place in the southern hemisphere. The belt of doldrums begins its retreat across the equator in January and continues moving north until June. It is within these months, therefore, that cyclones occur in the South Pacific and South Indian oceans. The more vigorous S. E. trade overtakes the slowly retreating N. W. trade and a cyclone is apt to result wherever their edges touch. The greater amount of land masses about the north Atlantic, resulting in a higher temperature potential for the equatorial circulation, as well as the fact that the doldrums do not cross the equator, seems to explain the greater severity of Atlantic tropical cyclones as compared with cyclones in other parts of the world.

It may be asked why the trades meeting obliquely should always set up a cyclonic whirl. It would seem as likely that anticyclonic whirls should result as cyclonic. The answer is that such whirls do occur, but since the deflective force and the pressure gradient are here in the same direction, they quickly collapse or shut up. In a cyclonic whirl, however, these forces are opposed and the tendency is to open out and continue.

The dynamic origin of whirls is a matter of every-day observation. A current of air meeting with any resisting object is deflected or reflected onto itself invariably setting up whirls. The wind whistling around a house, a post, a fence or any other obstacle continually gives rise to eddies which we recognize when they snatch up light objects. A current flowing over a sharp edge is always full of eddies on the leeward side. The Rocky Mountains

on the North American and the Andes on the South American continent, serving as barriers to half the atmosphere in the middle circulation, set up whirls on the leeward side, as the westerlies sweep against and over them. This is especially the case in winter, when the energy of this circulation, owing to the increased temperature gradient of the equatorial circulation, is about four times that of summer. Over the middle latitudes of the North American continent, whirl after whirl of cyclonic character reels off from the comb of the Rockies, every winter storm being first noted as a depression forming just to the leeward of this ridge at a latitude where the upper northwesterlies blow most fiercely. Certain it is that if the North American continent were flat, its weather would lose much of its cyclonic character and consist chiefly of monotonous "Brave West Winds." On weighing the two theories, therefore, we see that there is little to commend the local heat theory upheld by Ferrel and others, while, on the other hand, there is much in support of the dynamical origin of cyclones.

We have next to consider how a cyclone once formed perpetuates itself. By either theory the same kind of a circulation results, i.e., the air spirals in towards the center along the lower levels and spirals out again at the top. In rising through the center it expands approximately adiabatically and hence condensation must result. In a tropical hurricane the air is warm and saturated, containing a maximum amount of water. The vapor contained in the air is $\frac{1}{25}$ of it by volume, and $\frac{1}{37}$ by weight for 30° C. Since we may suppose the air to undergo a practically adiabatic expansion on rising, the temperature falls at a faster rate than the normal atmospheric rate. At no great height practically all of the vapor is condensed. The amount of latent heat evolved by this condensation is very great. We may get an idea of what it is from the following table.

LATENT HEAT OF VAPORIZATION AT DIFFERENT TEMPERATURES

Centigrade	Calories	Centigrade	Calories
0°	606	60°	565
10	600	70	558
20	593	80	551
30	586	90	544
40	579	100	537
50	572		

The condensation of the amount of vapor we have considered would give out heat enough to raise an equal weight of air 2467° C., and it would raise thirty-seven times its weight of air, or the air containing it, 66° C. This would expand it at constant pressure by nearly $\frac{1}{4}$.

We thus get a general idea of the enormous heating effect of the condensation of the vapor and the consequent very considerable expansion of the air in the axis of the cyclone. Once started, therefore, and given plenty of vapor we can easily see how the cyclone will persist or increase while liberating an enormous amount of energy. An ordinary heat engine destroys coal, thereby liberating energy and performing work. A cyclone is equally a heat engine, only it destroys vapor, in this manner liberating an enormous amount of energy and performing work.

We have now started our cyclone and provided it with means for its continuation. It behooves us to consider next what kind of forces it brings into play, and what forces oppose it.

If we regard only the rotatory velocities, neglecting the radial velocities, or, in other words, consider only the components $\Sigma r \dot{\varphi}$ of the cyclonic circulation, where $\dot{\varphi}$ is the angular velocity of a particle about the center and r its distance, neglecting the components $\Sigma \dot{r}$ which eliminate themselves, we shall have at any time a certain definite rotatory energy which we can replace by a solid rotating mass of properly selected dimensions and angular velocity. The masses of air circulating around the center of a cyclone represent gyratory energy just as much as a rotating flywheel or a liquid vortex. In the case of a gyrating solid the particles are held in a fixed relation to each other by the forces of cohesion. In a fluid vortex the particles are continually changing their relative position, but the congeries of positions as a whole does not change. That is to say, while in a rotating solid a certain velocity at a certain point is represented by the same set of particles in endless sequence, always preserving their mutual distances, in a rotating fluid we have the same velocity represented at the same point, but by a continually new set of particles.

The dynamical results are the same in either case. The stream lines being held in a fixed relation to each other by the forces we have considered, the vortex possesses a certain property of solids, viz., shape, and thus in a certain manner imitates a solid. The principle that fluids in motion may imitate solids has long been recognized and is the foundation of Lord Kelvin's vortex theory of atoms.

Since we have in a cyclone a rotating mass of air, and this mass, at the same time that it is rotating about its axis, is being carried around the axis of the earth, the combined system represents a gyroscope. For a gyroscope may be defined as a mass having simultaneous rotations about two different axes. Since, therefore, a cyclone is a gyroscope, we shall have to consider carefully the properties of such a doubly-rotating body. A complete discussion of the gyroscope will be found in the Appendix.

We shall define that portion of a spherical shell cut out by a small circle, as a spherical cap. In the present discussion the space occupied by a cyclone is a spherical cap. That is, we shall consider this space to be cir-

cular, although in reality it is, as a rule, only approximately so. In a spheroid, like the earth, we have seen that the component of gravitation along the surface always urges a body towards the pole. For a state of equilibrium this must be balanced by the component of centrifugal force acting towards the equator, or the gravitational component is equal to $R \sin \vartheta \cos \vartheta \, \omega^2$.

If we suppose the earth at rest and a spherical cap resting on its surface, it would, if there were no friction, slide towards the pole. If the earth were rotating, a component of centrifugal force, acting at the center of the cap, would urge it towards the equator. If, now, we set the cap rotating about its center, still other centrifugal forces will be developed. These are the centrifugal forces arising from the rotational velocities of the polar and equatorial halves of the cap. Although these velocities are in opposite directions, the centrifugal force of each acts towards the equator.

We shall call these two sets of centrifugal forces the revolutional and the rotational centrifugal forces.

It is further evident that such a rotating cap, which at the same time that it is rotating about its center is being carried around the axis of the earth, is a gyroscope. In the case of a cyclone, therefore, there are four independent forces acting along a meridian. The component of gravity and the gyroscopic force both urge it towards the pole, while the two centrifugal forces, viz., the revolutional and the rotational, urge it towards the equator.

We have seen that acting on each particle is a force $R \sin \vartheta \cos \vartheta \, (\omega^2 - \dot{\psi}^2)$, which urges it towards or away from the pole, according as its component of centrifugal force is in defect or excess of the gravitational component at the corresponding point of the earth's surface. The summation of all these forces for all the particles of a cyclone must be the equivalent of the four meridional forces we have just considered and must be capable of being resolved into these four forces.

The proof is as follows. Let us first consider two points, one on the extreme northern, the other on the extreme southern edge of the cyclone. If ρ be the distance from the center to these points and $\dot{\varphi}$ their angular velocity of rotation, the horizontal velocity of these points will be respectively

$R \cos \left(\vartheta_c + \frac{\rho}{R}\right) \dot{\psi}_c - \rho \, \dot{\varphi}$ and $R \cos \left(\vartheta_c - \frac{\rho}{R}\right) \dot{\psi}_c + \rho \, \dot{\varphi}$, where ϑ_c and $\dot{\psi}_c$ denote the latitude and horizontal angular velocity of the center.

If $\dot{\psi}$ denote the resultant horizontal angular velocity of a point, and ϑ its latitude, which is $\vartheta_c \pm \frac{\rho}{R}$, we have for the upper point,

$$v_h = R \cos \left(\vartheta_c + \frac{\rho}{R}\right) \dot{\psi}_c - \rho \, \dot{\varphi} = R \cos \vartheta \, \dot{\psi},$$

$$\text{or } \dot\psi = \frac{R\cos\left(\vartheta_c + \frac{\rho}{R}\right)\dot\psi_c - \rho\,\dot\varphi}{R\cos\vartheta}.$$ For the lower point

$$\dot\psi = \frac{R\cos\left(\vartheta_c - \frac{\rho}{R}\right)\dot\psi_c + \rho\,\dot\varphi}{R\cos\vartheta}.$$

Now the resultant polar force at any point is $R\sin\vartheta\cos\vartheta\,(\omega^2 - \dot\psi^2)$. Substituting, we have for the upper point

$$R\sin\left(\vartheta_c + \frac{\rho}{R}\right)\cos\left(\vartheta_c + \frac{\rho}{R}\right)$$

$$\left[\omega^2 - \left(\frac{R^2\cos^2\left(\vartheta_c + \frac{\rho}{R}\right)\dot\psi_c^2 - 2\,R\,\rho\,\dot\varphi\,\dot\psi_c\cos\left(\vartheta_c + \frac{\rho}{R}\right) + \rho^2\,\dot\varphi^2}{R^2\cos^2\left(\vartheta_c + \frac{\rho}{R}\right)}\right)\right], \quad (1)$$

and for the lower point.

$$R\sin\left(\vartheta_c - \frac{\rho}{R}\right)\cos\left(\vartheta_c - \frac{\rho}{R}\right)$$

$$\left[\omega^2 - \left(\frac{R^2\cos^2\left(\vartheta_c - \frac{\rho}{R}\right)\dot\psi_c^2 + 2\,R\,\rho\,\dot\varphi\,\dot\psi_c\cos\left(\vartheta_c - \frac{\rho}{R}\right) + \rho^2\,\dot\varphi^2}{R^2\cos^2\left(\vartheta_c - \frac{\rho}{R}\right)}\right)\right]. \quad (2)$$

From (1) we have

$$R\,\ddot\vartheta = R\sin\vartheta\cos\vartheta\,(\omega^2 - \dot\psi_c^2) + 2\,\rho\,\dot\varphi\sin\left(\vartheta_c + \frac{\rho}{R}\right)\dot\psi_c - \frac{\rho^2\,\dot\varphi^2}{R}\tan\left(\vartheta_c + \frac{\rho}{R}\right).$$

From (2 we have)

$$R\,\ddot\vartheta = R\sin\vartheta\cos\vartheta\,(\omega^2 - \dot\psi_c^2) - 2\,\rho\,\dot\varphi\sin\left(\vartheta_c - \frac{\beta}{R}\right)\dot\psi_c - \frac{\rho^2\,\dot\varphi^2}{R}\tan\left(\vartheta_c - \frac{\rho}{R}\right).$$

Adding, we have

$$R\,\ddot\vartheta = R\,(\omega^2 - \dot\psi_c^2)$$

$$\left[\sin\left(\vartheta_c + \frac{\rho}{R}\right)\cos\left(\vartheta_c + \frac{\rho}{R}\right) + \sin\left(\vartheta_c - \frac{\rho}{R}\right)\cos\left(\vartheta_c - \frac{\rho}{R}\right)\right]$$

$$+ 2\,\rho\,\dot\varphi\,\dot\psi_c\left[\sin\left(\vartheta_c + \frac{\rho}{R}\right) - \sin\left(\vartheta_c - \frac{\rho}{R}\right)\right]$$

$$- \frac{\rho^2\,\dot\varphi^2}{R}\left[\tan\left(\vartheta_c + \frac{\rho}{R}\right) + \tan\left(\vartheta_c - \frac{\rho}{R}\right)\right].$$

Developing and reducing,

$$R\ddot{\vartheta} = R\,(\omega^2 - \dot{\psi}_c^2)\; 2 \sin\vartheta_c \cos\vartheta_c \cos\frac{2\rho}{R} + 4\rho\,\dot{\varphi}\,\dot{\psi}_c \cos\vartheta_c \sin\frac{\rho}{R}$$

$$-\frac{2\rho^2\dot{\varphi}^2}{R}\tan\vartheta_c \frac{\sec^2\frac{\rho}{R}}{1-\tan^2\vartheta_c\tan^2\frac{\rho}{R}}. \quad (3)$$

The first term in (3) represents the difference between the component of gravitation applied at the center and the revolutional centrifugal component applied at the same point. The second term is the gyroscopic force and the last term the rotational centrifugal component. Taking a complete circle of points at the distance ρ from the center we see that for the east and west points the first term in (3) becomes $R\,(\omega^2 - \dot{\psi}_c^2)\; 2 \sin\vartheta_c \cos\vartheta_c$, while the other two terms vanish. We may write then as an approximate average for the circle,

$$R\ddot{\vartheta} = R\,(\omega^2 - \dot{\psi}_c^2)\; 2 \sin\vartheta_c \cos\vartheta_c \frac{1+\cos\frac{2\rho}{R}}{2} + 2\rho\,\dot{\varphi}\,\dot{\psi}_c \cos\vartheta_c \sin\frac{\rho}{R}$$

$$-\frac{\rho^2\dot{\varphi}^2}{R}\tan\vartheta_c \frac{\sec^2\frac{\rho}{R}}{1-\tan^2\vartheta_c\tan^2\frac{\rho}{R}}. \quad (4)$$

Since the angle $\frac{\rho}{R}$ is generally small, especially in tropical hurricanes in their early course, we may consider $\cos\frac{\rho}{R}$ and $\sec\frac{\rho}{R}$ as practically unity, while $\sin\frac{\rho}{R}$ is nearly $\frac{\rho}{R}$. Consequently, we can write (4).

$$R\ddot{\vartheta} = 2R\,(\omega^2 - \dot{\psi}_c^2)\sin\vartheta_c\cos\vartheta_c + \frac{2\rho^2\dot{\varphi}\dot{\psi}_c}{R}\cos\vartheta_c - \frac{\rho^2\dot{\varphi}^2}{R}\tan\vartheta_c. \quad (5)$$

Or, dividing by the mass,

$$R\ddot{\vartheta} = R\,(\omega^2 - \dot{\psi}_c^2)\sin\vartheta_c\cos\vartheta_c + \frac{\rho^2\dot{\varphi}}{R}\dot{\psi}_c\cos\vartheta_c - \frac{\rho^2\dot{\varphi}^2}{2R}\tan\vartheta_c. \quad (6)$$

We have so far only considered the rim of the cyclone. Since the average value of ρ is the radius of gyration, which we shall denote by k, we may write for the whole cyclone,

$$R\ddot{\vartheta} = R\sin\vartheta_c\cos\vartheta_c\,\omega^2 - R\sin\vartheta_c\cos\vartheta_c\,\dot{\psi}_c^2 + \frac{k^2\dot{\varphi}}{R}\dot{\psi}_c\cos\vartheta_c - \frac{k^2\dot{\varphi}^2}{2R}\tan\vartheta_c. \quad (7)$$

The terms are now in an easily recognizable form. $R \sin \vartheta_c \cos \vartheta_c \omega^2$ is the gravitational component, the plus sign showing that it acts towards the pole.

$R \sin \vartheta_c \cos \vartheta_c \dot{\psi}_c^2$ is the revolutional centrifugal component, the minus sign showing that it acts towards the equator. $\frac{k^2 \dot{\varphi}}{R} \dot{\psi}_c \cos \vartheta_c$ is the expression for the gyroscopic force with which we are already familiar. (See Appendix.) The plus sign shows that it acts towards the pole. $\frac{k^2 \dot{\varphi}^2}{2 R} \tan \vartheta_c$ is the rotational centrifugal component, the minus sign showing that it acts towards the equator.

The rotational centrifugal force itself, that is not resolved along the surface, is $-\frac{k^2 \dot{\varphi}^2}{2 R \cos \vartheta_c}$.

$k^2 \dot{\varphi}^2$ is the square of the velocity of the whole mass concentrated in a ring at the distance k from the center, while $R \cos \vartheta_c$ is the distance of the center from the axis of the earth, thus showing that it is a centrifugal force.

We thus see that the deflective forces of all the points of a cyclone, when summed together, are equivalent to the four forces we have just discussed, acting at the center of the cyclone. Of these four forces the two centrifugal forces act towards the equator, while the gravitational and gyroscopic forces act towards the pole. When these forces are not in equilibrium they will urge the cyclone either north or south—generally north, in the northern hemisphere—until a position of equilibrium is found, and there will always be some latitude between the equator and the pole where this equilibrium is attained.

Equation (6), which we may write

$$R \ddot{\vartheta} = A \sin \vartheta \cos \vartheta \left(\omega^2 - \dot{\psi}^2\right) + B \cos \vartheta \dot{\psi} - C \tan \vartheta \qquad (8)$$

cannot be solved until we determine the manner in which the fifth force, viz., friction, acts upon the cyclone.

This force has evidently no effect in moving the cyclone in a meridional direction, but it exerts a moment tending to move it about the axis of the earth in an anticyclonic direction, i.e., from east to west.

If we suppose an arm $A B$ pivoted to a table at A and to a disc at B, its center, so that the disc can turn about this center while lying on the table and the whole arrangement is capable of turning about A, we have, after a fashion, a frictional model of a cyclone. If the disc is turning in a counterclockwise direction, the frictional forces act upon the lower half in the direction indicated by the arrow, or from east to west, while the frictional

forces on the upper half act in a contrary direction. For equilibrium the upper frictional forces must be greater than the lower, since they act upon a shorter arm.

We shall neglect the friction due to the motion of the cyclone as a whole over the surface of the earth, as this velocity is much less than the rotational velocity. It is probable that the friction of the air on the earth's surface is a function that increases rapidly with the velocity—at least as the square of the velocity. Hence, the rotational friction arising from a velocity of 100 to 150 miles an hour will far outweigh the translational friction due to a velocity of 10 to 20 miles on an average. Hence it seems allowable to neglect the latter in comparison with the former, and an examination of actual cyclone paths appears to bear this out.

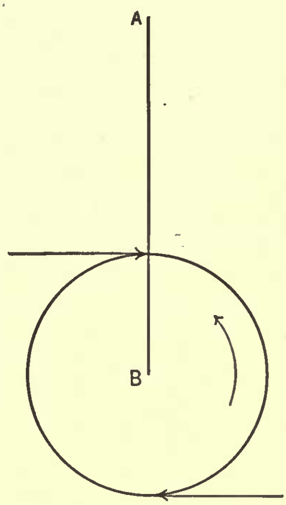


FIG. 20

If, now, we suppose a cyclone held at a certain latitude by the opposing polar forces we have just considered, and which, in this sense, take the place of the arm $A B$ in Fig. 20, we shall have motion about the point A if the moments of the two opposing frictional forces are not equal. This motion is almost always, at first, in an anticyclonic direction, since the frictional forces on the lower half act on a longer arm. There is, however, a tendency for these two moments to become equal, and in actual cyclones this appears to be effected in a short time. This is probably brought about by a general deformation of the cyclone, the flow of the air in the equatorial half being impeded by the increased resistance and giving the air in the polar half an opportunity to spread out over a greater surface, and thus by an increased frictional area to bring its moment up to that of the lower half. The actual facts are that a cyclone preserving the same latitude always moves with a constant horizontal velocity, and in moving from a lower to a higher latitude a cyclone always decreases its horizontal velocity. No example has ever been found of a cyclone remaining on the same parallel and varying its horizontal velocity, nor any example of a cyclone moving towards the pole and not diminishing its horizontal velocity. An example of an actual cyclone moving with a constant latitude is given in Fig. 21, A. Under such conditions all the forces must be in equilibrium, and a cyclone performing such a motion may be called a Poinsot cyclone. For a body under the action of no forces performs the motions we have already investigated in the Appendix, viz., a Poinsot motion, where a mass rotates about two different axes simultaneously under the action of no forces.

Let us suppose that in a cyclone, in a certain latitude, the frictional

forces on the upper half, which tend to move the cyclone in a cyclonic direction, are in equilibrium with the frictional forces on the lower half, which tend to move it in an anticyclonic direction. If now the cyclone moves to a higher latitude, these frictional forces will no longer be in equilibrium, since the anticyclonic moment will preponderate over the cyclonic moment. In other words, the equilibrium has been disturbed and the preponderant frictional forces on the equatorial side will move the cyclone from east to west and cause a reduction of the horizontal velocity. We are speaking of the absolute horizontal velocity of the cyclone, *not* that relatively to the earth.

We see, then, that a cyclone cannot move towards the pole without losing some of its absolute horizontal velocity, while conversely it cannot move towards the equator without gaining absolute horizontal velocity. As it moves towards the pole an excess moment comes into play which we may consider the equivalent of a force f, acting to the westward, into an arm $R \operatorname{ctn.} \vartheta$. If we suppose this excess moment to remain constant from point to point as the cyclone progresses to the north, then the force f will be inversely proportional to the arm or as the tangent of the latitude. The amount of the reduction of the horizontal velocity will, therefore, be greater the nearer the cyclone approaches to the pole.

Once the cyclone has arrived at its latitude of equilibrium, the excess moment quickly vanishes and the motion becomes uniform. Since this excess moment only exists temporarily while the cyclone is moving from point to point towards the pole, we may consider that the reduction of the horizontal velocity is proportional to the force f. Consequently if v_o be the proper horizontal velocity for the cyclone in latitude ϑ_o, then in latitude ϑ it will be $v_o - K \tan \vartheta$, or $v_h = v_o - K \tan \vartheta$, or generally $v_h = C - K \tan \vartheta$, where C and K are some constants to be determined. We shall call this the tangent law of the cyclone.

We have departed in the above somewhat from our former strict methods. Such reasoning is far from strict, but merely plausible. Under the circumstances it seems hardly possible that the problem of friction can be subjected to strict analytical methods. The cyclone is continually altering its shape and the configuration of its stream lines, this altering and readjustment being itself an indication of the frictional forces at work. However, our reasoning, though only plausible, will be justified if it leads to some practical result. To judge of this, we shall have to apply it to some actually charted cyclones.

It may happen, and frequently does, that where a cyclone is changing its latitude rapidly, the excess frictional moment in the anticyclonic direction does not have time to reduce the horizontal velocity to the value indicated by our law. In such cases we shall find a slight discrepancy between the

calculated and observed values, ranging from 1 to 4 miles, and always in excess of the calculated value.

In Fig. 21 are given three cyclones charted by Reid and Redfield. It will be seen that cyclone A is executing a Poinsot motion; that is, it preserves its latitude as well as a constant horizontal velocity. Such cyclones are rare in low latitudes, as the gyroscopic forces are almost always in excess of the centrifugal.

For cyclone B, we have the following data. We shall use in these calculations geographical miles instead of statute miles as heretofore.

On the 13th, it was in Long. 66.75° W. On the 14th, it was in Long. 73.25°. The middle latitude was 19° N. Since a degree of longitude at Lat. 19° contains 56.8 geographical miles, and it had traveled 6.5° in 24 hours, its velocity relatively to the earth was 15.5 miles. Since the velocity of the earth here is 852.1 miles, its absolute horizontal velocity was 836.6 miles.

Similarly, between the 17th and 18th, it had moved over 4.25° of longitude with an average latitude of 27°. Since a degree at this latitude contains 53.5 geographical miles, it had moved over 227.6 miles relatively to the earth in 24 hours, or its relative horizontal velocity was 9.5 miles. The velocity of the earth at this latitude is 803.2 miles. Hence, the absolute horizontal velocity was 793.7 miles. Substituting these values in our formula, we have

$$836.6 = C - K\,.34433$$
$$793.7 = C - K\,.50953$$

By elimination we find that $C = 927.2$ and $K = 262.5$.

From these parameters we can construct the following table.

ABSOLUTE HORIZONTAL VELOCITY

Date	Observed	Calculated
13	836.6	
14	820.8	821.2
15	814	813
16	805.7	804.8
17	793.7	
18		

For the cyclone C, we find it at Lat. 15°, Long. 77° W., on the 26th September. On the 27th it was in Lat. 16° 30′ and Long. 79°. The average latitude between these two positions is 15° 45′. A degree of longi-

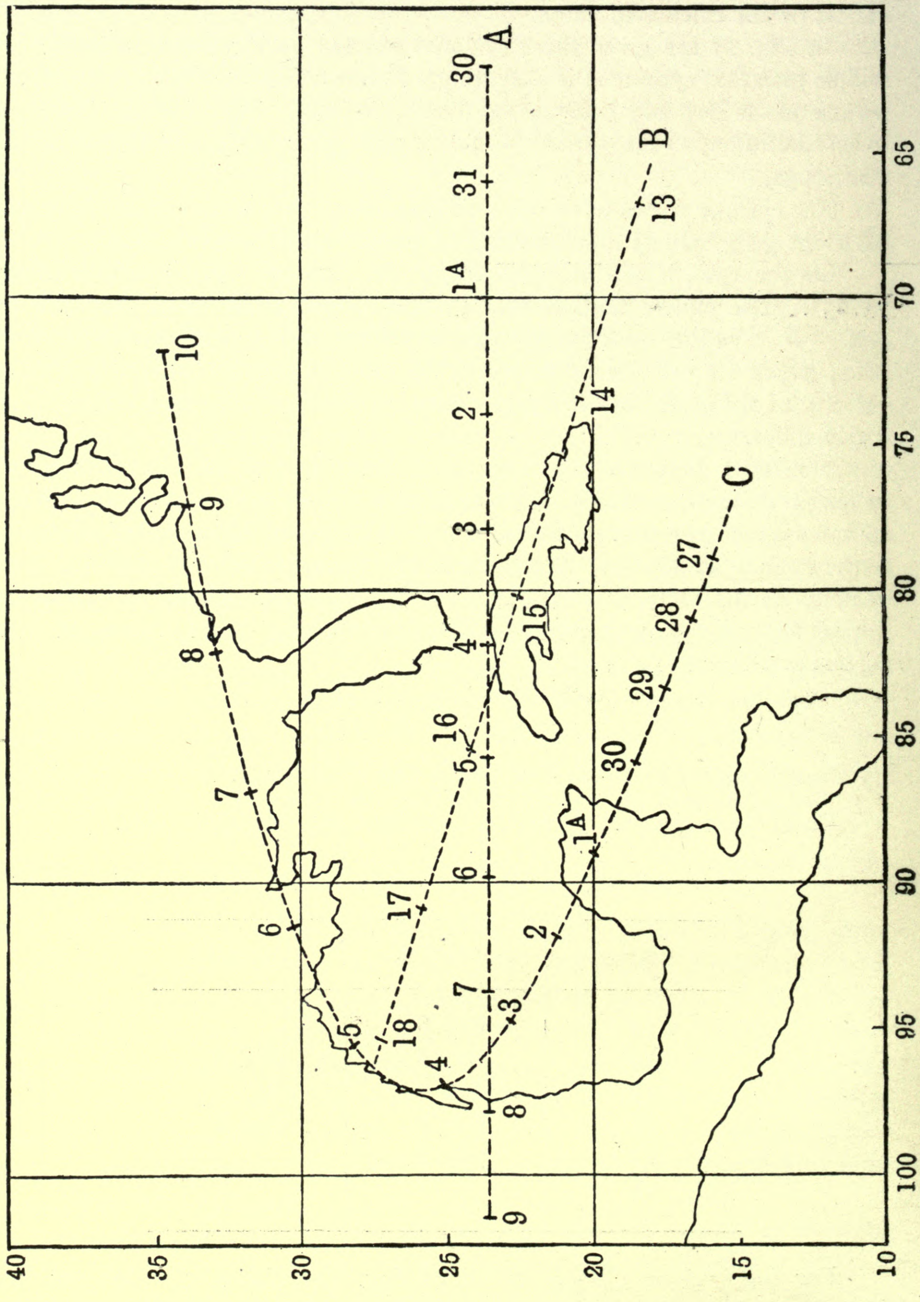


FIG. 21

tude at this latitude contains 57.8 miles. Hence, there was a relative velocity of 4.8 miles and an absolute horizontal velocity of 862.6 miles. From the 9th to 10th October it moved 5.5° of longitude in an average latitude of 34°. Since a degree of longitude here is 49.851 miles, it moved with a relative velocity of 11.4 miles and with an absolute horizontal velocity of 758.8 miles.

Substituting in our formula, we have

$$862.6 = C - K\ .28203$$
$$758.8 = C - K\ .67451$$

Whence we find that $C = 936.5$ and $K = 262.2$. We can thus construct the following table.

CYCLONE C. HORIZONTAL VELOCITY

Latitude	Observed	Calculated
15° 45′	862.6	
17°	856.9	856.4
17° 30′	853.4	853.8
18°	851.1	851.3
19°	845	846.2
21°	835.1	835.9
22°	829.4	830.6
23° 45′	820.4	821.2
26° 30′	806.5	805.7
28° 30′	797.3	794.2
31° 30′	778.8	776
32°	774.8	772.7
33°	766.8	766.1
34°	758.8	

It will be noted that the actual values slightly exceed the calculated ones where the cyclone was running over the land.

We shall next consider the Porto Rican hurricane of August, 1899. On the 7th it went from Long. 60° 30′ W. to Long. 66° W. in the average latitude 16° 50′. On the 8th it went from Long. 66° to Long. 70° 30′ in an average latitude 18° 45′. Since a degree of longitude is 57.5 miles at Lat. 16° 50′, its relative velocity was 13.1 miles, and its absolute horizontal velocity 851.3 miles. Since a degree of longitude, at 18° 45′, is 56.9 miles, its relative velocity here was 10.7 miles and its absolute horizontal velocity 842.6 miles.

Substituting in our formula,

$$851.3 = C - K\ .30255$$
$$842.6 = C - K\ .33945$$

Whence we find that $C = 923$ and $K = 236$. To find the point of recurvation we must solve the equation $v_e = 900 \cos \vartheta = 923 - 236 \tan \vartheta$, where v_e represents the velocity of the earth. That is to say, the point of recurvation will be where the earth and the cyclone are moving with the same velocity. Lat. 28° gives a velocity of 797 miles for the cyclone, and this is about the velocity of the earth at this point. Consequently the path will recurve near the parallel of 28°.

It must be remembered that the data derived from the chart are not strictly accurate and small errors, where the two positions from which the constants are calculated are near together, will make appreciable errors in these constants. Taking the more probable values $C = 927$ and $K = 250$, we get the following table:

PORTO RICAN HURRICANE OF AUGUST 8, 1899
HORIZONTAL VELOCITY

Latitude	Observed	Calculated
16° 50′	851	851.4
17°	849	850.4
18°	845.8	845.8
19°	841.4	841.0
19° 30′	839.2	838.1
20° 30′	837.2	833.8
21°	835.1	831.3
22° 30′	826.2	823.5
24° 30′	814.5	813.5
26°	805.8	805.3
27°	799.2	799.7
28°	794	794.1
29°	788	788.5
30°	782	782.8

We see both from the calculated and observed positions that the cyclone did not recurve sharply, but remained practically on the same meridian from Lat. 26° to 30°. That is, it so happened that the decrease of the horizontal velocity from friction nearly kept step with the decrease of the earth's velocity from the 26th to the 30th parallel.

Another hurricane given by Piddington had the following data.

Date	Latitude	Longitude
Oct. 11	25°	82° 30′ W
Oct. 12	31° 15′	82° 15′ W
Oct. 13	38° 20′	78° W
Oct. 14	47° 15′	68° 30′ W

$C = 953$
$K = 295.4$

Taking the average latitudes we have

HORIZONTAL VELOCITY

Date	Latitude	Observed	Calculated
11th to 12th	28°	796	796
12th to 13th	34° 45′	749.7	748
13th to 14th	42° 45′	680	680

This cyclone went rapidly to the north and must have possessed great gyroscopic force and, therefore, great energy. The preceding cyclones, without multiplying examples, show that the tangent law fits the facts rather closely. The law has been applied in a number of other cases with the same result. No contradiction has been met with.

Whether we regard this law as a plausible deduction, or merely as an empirical formula, in either case it supplies us with the necessary material for the completion of our calculation. We already have

$$R\ddot{\vartheta} = R \sin\vartheta \cos\vartheta\, \omega^2 - R \sin\vartheta \cos\vartheta\, \dot{\psi}^2 + \frac{k^2 \dot{\varphi}}{R} \cos\vartheta\, \dot{\psi} - \frac{k^2 \dot{\varphi}^2}{2R} \tan\vartheta. \quad (8)$$

And the tangent law which we may write,

$$v_h = R \cos\vartheta\, \dot{\psi} = a - b \tan\vartheta.$$

$$\therefore \dot{\psi} = \frac{a}{R} \sec\vartheta - \frac{b}{R} \tan\vartheta \sec\vartheta. \quad (9)$$

Substituting (9) in (8)

$$R\ddot{\vartheta} = R \sin\vartheta \cos\vartheta\, \omega^2 - \frac{a^2}{R} \tan\vartheta + \frac{2ab}{R} \tan^2\vartheta - \frac{b^2}{R} \tan^3\vartheta$$
$$+ \frac{k^2 \dot{\varphi} a}{R} - \frac{k^2 \dot{\varphi} b}{R} \tan\vartheta - \frac{k^2 \dot{\varphi}^2}{2R} \tan\vartheta.$$

Multiplying by $\dot{\vartheta}$ and integrating,

$$v_p^2 = -\frac{R^2\,\omega^2}{2}\cos 2\,\vartheta + 2\,(a^2 - b^2)\log\cos\vartheta + 4\,a\,b\tan\vartheta$$

$$-\,b^2\tan^2\vartheta - \left(4\,a\,b - \frac{2\,k^2\,\dot{\varphi} a}{R}\right)\vartheta + \left(\frac{2\,k^2\,\dot{\varphi}\,b}{R} + k^2\,\dot{\varphi}^2\right)\log\cos\vartheta + K. \quad (10)$$

We can reduce this to

$$v_p^2 = -\frac{R^2\,\omega^2}{2}\cos 2\,\vartheta + 4\,a\,b\tan\vartheta - b^2\tan^2\vartheta$$

$$-\,D\,\vartheta + E\log\cos\vartheta + K \qquad (11)$$

or generally,

$$v_p^2 = -\,A\cos 2\,\vartheta + B\tan\vartheta - C\tan^2\vartheta - D\,\vartheta + E\log\cos\vartheta + K, \qquad (12)$$

where the capitals are constants in general.

In order to determine the constants in Equation (12) it would be necessary to observe the polar velocities at six points, although theoretically Equation (10) would only require three points. These equations are utterly unmanageable in practice.

However, they may be reduced by the aid of series. Since a function can be expressed as a series by the aid of Maclaurin's theorem, which is

$$f(\vartheta) = f(o) + \vartheta f'(o) + \frac{\vartheta^2}{2\,!} f''(o) + \frac{\vartheta^3}{3\,!} f'''(o) \text{ etc.,}$$

we can develop the functions in Equation (12) into the following forms, by neglecting powers of ϑ above the third.

$$\cos 2\,\vartheta = 1 - 2\,\vartheta^2. \qquad \tan\vartheta = \vartheta + \frac{2\,\vartheta^3}{3\,!}.$$

$$\tan^2\vartheta = \frac{2\,\vartheta^3}{3\,!}. \qquad \log\cos\vartheta = -\frac{\vartheta^2}{2}.$$

Substituting these expressions in Equation (12) we have,

$$v_p^2 = -\,A\,(1 - 2\,\vartheta^2) + B\left(\vartheta + \frac{2\,\vartheta^3}{3\,!}\right) - C\left(\frac{2\,\vartheta^3}{3\,!}\right) - D\,\vartheta - E\,\frac{\vartheta^2}{2} + K. \quad (13)$$

Reducing and again letting capitals express constants in general (not the same as before), we have

$$v_p^2 = A\,\vartheta - B\,\vartheta^2 + C\,\vartheta^3 - K. \qquad (14)$$

The values of ϑ are, of course, expressed here in radians. Since a radian is 57.295°, Equation (14) only holds approximately for low latitudes. When ϑ approaches unity, the series becomes less convergent and above unity the series is divergent. Where we limit ourselves to the third power of ϑ it would probably be unsafe to use the formula for latitudes above 40°. If the latitudes are small, not above 30°, we may with some degree of approximation omit the term containing the cube and write

$$v_p^2 = A\,\vartheta - B\,\vartheta^2 - \mathrm{K}. \qquad (15)$$

As an example we shall take the Porto Rican hurricane which we have already discussed.

In Lat. 16° this cyclone had a polar velocity of 5 miles an hour.

In Lat. 18° it had, as nearly as we can make out, the same polar velocity.

In Lat. 19.25° the polar velocity was 4.5 miles.

From these three points we can calculate the constants in Equation (15). Whether we write the latitude in radians or degrees is immaterial, except that it gives us different values for the constants. We can, therefore, write the three equations, using degrees,

$$\begin{aligned} 25 &= A\,16 - B\,256 - K \\ 25 &= A\,18 - B\,324 - K \\ 20.25 &= A\,19.25 - B\,370.56 - K \end{aligned}$$

Eliminating we find that

$$\begin{aligned} A &= -\ 12.47 \\ B &= -\ .365 \\ K &= -\ 131.08 \end{aligned}$$

Substituting the value of these constants in Equation (15), we find that for latitude 25°, $v_p = \sqrt{47.5}$, which agrees with what was actually observed, viz., a velocity somewhat less than seven miles an hour.

At 35° the polar velocity was somewhat less than 12 miles an hour, and as far as we can judge it increased rapidly as it got farther north, though no record of its subsequent course is available. Our formula (15) does not carry us with any safety above 30°.

Again, for cyclone *C*, which we have already considered, we find that

$$\begin{aligned} A &= 28.3 \\ B &= .56 \\ K &= 308. \end{aligned}$$

This shows a polar velocity of 1 mile an hour at 16°, 7 miles an hour at 25°, and zero velocity at 35°, and for higher latitudes the polar velocities are imaginary. Actually it ceased northing in about 36°.

These results are, of course, only approximate. It is not to be expected that we can calculate the path of a cyclone with the same exactness that we do that of a heavenly body. By the aid of these formulas, however, we can acquire a very good idea of the general shape of the path. We can generally determine about where it will cease northing or where it attains its latitude of equilibrium, where it will recurve and block out roughly its general path. By path is, of course meant the path of the axis. Since the cyclone extends usually a hundred or more miles on all sides of the axis, this knowledge, although not exact, will be useful. In the case of a tropical hurricane, therefore, it will be possible to predict the weather a week or

more ahead. Incidentally, these results are of value in confirming our theory. They show that the motions of cyclones are practically what we had reason to believe must result from the action of the forces at work.

Ferrel attempted to explain the northing of cyclones in the following manner. He says ("Motions of Fluids and Solids Relative to the Earth"): "Now these deflecting forces being as the sine of the latitude, the pressure on the polar side towards the pole is greater than that on the other side towards the equator, and hence the cyclone moves in the direction of greatest pressure." So far as the latitude alone is concerned, this is true; but the deflective force is $2 V \omega \sin \vartheta$, and depends also upon the factor V, which may be greater in a moving cyclone on the equatorial side than on the polar side. According to Ferrel's explanation, a cyclone should move continually towards the pole until it reaches it, which we know is impossible. The polar forces urging a cyclone are, as we have seen, a conbination of four forces, gyroscopic, centrifugal and gravitational, which are eventually derived from the deflective forces. These forces at first usually urge the cyclone towards the pole, but after a time an equilibrium occurs between these four forces, after which the cyclone remains upon its parallel of equilibrium, or rather oscillates about this parallel of equilibrium.

Meteorology being in its infancy, it has unfortunately been the custom to attempt to explain all its phenomena off hand, at a time when a lack of knowledge of the forces at work rendered such explanations impossible. The idea seems to have prevailed that some explanation was necessary, and that any explanation carried with it necessarily some advantage. At times a certain cause has been given for some occurrence and later on the same cause has been called upon to accomplish a directly opposite effect. We shall shortly give such an instance. In the movement of cyclones, the fact that a rotational frictional couple exists, tending usually to turn the cyclone about the axis of the earth in an anticyclonic direction has been ignored, as well as the fact that a cyclone is a gyroscope and that centrifugal and gravitational forces are at work.

Among various causes which have been assigned for the motions of cyclones we may mention the following. They have been said to be guided by and to follow up the Gulf Stream.

The most cursory examination of cyclone paths will show that this is certainly not the case. They have been thought to be guided by the coast line. No further answer to this is required than that it is evidently not so. It is true that in low latitudes the coast lines have a certain general resemblance to cyclone curves. It is possible that in remote times, when the constituents of the earth's crust were gaseous and formed a part of the atmosphere, the cyclones of that time precipitated cumulatively their contents along their paths and thus gradually built up the present shore configurations. The sun to-day is in such a condition and its atmosphere contains vortices or cyclones

which move along cyclone paths towards the pole. What are known as sun spots are in all probability cyclones, and their movement towards the poles is to be explained upon cyclonic (gyroscopic) principles. To say, however, that cyclones are guided by the present shore lines is an inversion of cause and effect. While it is possible that the present shore lines are the result of ancient cyclones, it is certain that the present cyclone paths are not governed by these shore lines. It is frequently stated in Weather Reports and books on Meteorology that cyclones are deflected by distant areas of high pressure. Here again the most cursory examination will convince us that such is manifestly not the case.

Lastly, the east-west motions of cyclones are said to be due to the general circulation of the atmosphere. Curiously the north-south components of the general circulation are supposed to have no effect on the motion of cyclones. This is Ferrel's explanation. According to him, cyclones should move towards the west while in the tropics because there is here a westerly component in the general circulation. That is to say, there is a westerly component in the lower strata—up to 15,000 ft. In the higher strata there are easterly components, but these must have no effect. Again, after passing the high pressure ridge (Lat. 34°) the general drift of the atmosphere is towards the east; hence, cyclones must move towards the east in these latitudes. True, at the surface these easterly winds are slight or wanting. However, in the upper strata this drift is very pronounced, and it is now the upper strata which propel the cyclone. Hence, these intelligent winds arrange it among themselves to guide the cyclone in its proper path. During its early course the task is allotted to the surface winds, while later on the higher currents assume all responsibilities. We might go on to inquire why these winds blow a cyclone at times so little to the west and then blow a second one very much farther to the west, although only a short time has elapsed between their passages. But it is needless to pursue the subject further. A careful examination of cyclone paths will show that their motions are practically independent of the general circulation.

A cyclone is a thin disc of some hundreds, perhaps thousands, of miles in diameter. It grips the earth by its frictional forces and presents only its thin edge to the general circulation. The general currents on striking this edge are immediately absorbed into the general whirl or they may even in the highest strata flow over the cyclone, but it is evident that they can have little, if any, effect in moving it. To this there is probably one exception. When the whirl is formed in the upper strata and has no connection with the earth, it is likely that it is carried along with the currents in which it was formed. The whirls formed on the leeward side of a high mountain comb which lies obliquely to a strong upper current, are in all probability carried along at first with the current until they finally take root in the ground. This would seem to be the case with the whirls which are reeled

off from the comb of the Rocky-Andes range, though even here gyroscopic forces will affect their course. Again, in the case of very limited whirls formed in the upper layers, such as tornadoes, these local manifestations must be carried along with the general currents, since they have no footing on the earth, or *pied à terre,* by which to hold themselves. But observation shows that well-defined cyclones, starting on the earth, such as the heavy tropical hurricanes, are practically uninfluenced by the general circulation.

TORNADOES

Tornadoes are cyclones of very limited extent formed at a considerable height above the surface of the earth. The rotation of these whirls is always cyclonic—counterclockwise in the northern hemisphere. The theory that this phenomenon is caused by the convectional ascent of a local column of heated air seems untenable for the reasons already given in the discussion of the formation of cyclones. As in the former case, it seems more probable that they are formed by the opposition of two currents meeting obliquely. In other words, they probably have a dynamical origin just as is the case with the larger cyclones. In the equatorial circulation we have seen that currents in nearly opposite directions are often overlying each other. In the seasonal shift of the circulations, north and south, it may be supposed that, during the process of readjustment, a lower stream may be pushed up against a higher stream blowing in a contrary direction at some point. In fact this must be frequently the case. The resulting whirl may be clockwise or counterclockwise. If the former, it will quickly close up, as we have seen, and its existence will be of short duration. If counterclockwise, it may persist provided it finds at hand the necessary fuel, or we may say, ammunition, in the form of water vapor.

In the middle circulation the same thing is liable to happen. The streams of the circulation flowing to the eastward are continually urged by the deflective force to turn to the right. They are kept in their course by the pressure gradient which opposes this motion and urges them towards the pole. When these forces are in equilibrium, they will continue their eastward motion. But it is easily conceivable that in the seasonal shifting of this circulation these forces may at some point suddenly fall out of equilibrium. If the deflective force should suddenly become greatly preponderant, the stream would in fact turn to the right and endeavor to short-circuit itself by turning in a circle. This, in all probability, frequently happens. If, now, the returning edge, which is moving to the westward, comes into opposition to the regular easterly drift at some point, a whirl will result. If, as we have pointed out, this whirl should be counterclockwise and found plenty of saturated or nearly saturated air at hand, then a tornado would result. There are some essential points of difference between these upper whirls when they are formed in the equatorial and when

they are formed in the middle circulation. Though having many points in common and essentially the same phenomena dynamically, what is known as a tornado is essentially a phenomenon of the middle circulation, while a waterspout is more a phenomenon of the equatorial circulation.

We must suppose that a tornado at its upper surface—its surface of formation—has a considerable area, perhaps many square miles. At its center its velocity is greatest, and hence its friction on the next underlying layers is here greatest. The whirl will propagate itself downward from the center in a peculiar funnel-shaped form. Its shape in fact will be like that of the water flowing out of a pipe at the bottom of a basin, for the following reasons. The centrifugal force due to the whirl must at every point be in equilibrium with the outside pressure of the general atmosphere. This pressure increases with the depth. Hence, the diameter of the whirl must decrease the lower it gets, since the centrifugal force varies inversely as this diameter. More strictly, the moment of the horizontal velocity is preserved as the center bores its way downward to the earth; whence the centrifugal force is inversely as the cube of the radius.

If we call the height of a point above the earth z, and r the radius of the horizontal section, we have $v = \frac{C}{r}$ and the centrifugal force is $\frac{C^2}{r^3}$.

Since the velocity would be infinite at the axis of the tornado, there cannot be any air there. In other words, a tornado is a hollow tube with an absolute vacuum along its axis. At any point of this sharply delimited surface, the forces urging a particle along a tangent to the surface in the plane of the axis, are $\frac{C^2}{r^3} \sin \varphi - g \cos \varphi$ (1), where φ is the inclination of this tangent to the axis of the tornado.

The upward component, due to the centrifugal force, is generally greater than the downward component due to gravity.

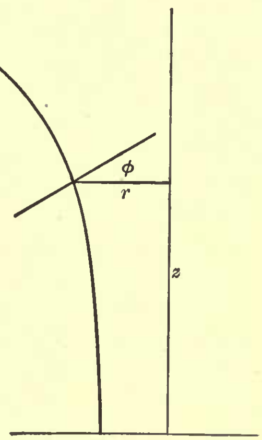


FIG. 22

Since $\sin \varphi = \frac{dr}{ds}$, and $\cos \varphi = \frac{dz}{ds}$, we can write (1)

$$\frac{C^2}{r^3} \cdot \frac{dr}{ds} - g\frac{dz}{ds} = \frac{d^2 s}{dt^2}. \qquad (2)$$

Multiplying by $\frac{ds}{dt}$, we have,

$$\frac{C^2}{r^3} \cdot \frac{dr}{dt} - g\frac{dz}{dt} = \frac{d^2 s}{dt^2} \cdot \frac{ds}{dt}.$$

Integrating, $$-\frac{C^2}{2r^2} - gz = \frac{1}{2}\left(\frac{ds}{dt}\right)^2 + K \qquad (3)$$

Calling r_0 the value of r at the surface of the earth, and v and v_0 the values of the velocity of a particle in the direction of the tangent to the surface at any point and at the surface of the earth respectively, we have

$$\frac{C^2}{2}\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) = gz + \frac{1}{2}(v^2 - v_0^2). \qquad (4)$$

This equation shows that the work of raising a particle to a height z against gravity, and of increasing its kinetic energy in a vertical plane from $\frac{v_0^2}{2}$ to $\frac{v^2}{2}$, is at the expense of the centrifugal energy.

If the particle which is whirled up by the tornado starts from rest at the surface of the earth, we can write Equation (4)

$$\frac{C^2}{2}\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) = gz - \frac{1}{2}v^2. \qquad (5)$$

Such a particle as we have imagined, therefore, upon this surface, not only moves about the axis, but shoots upwards. In other words, it spirals upwards rapidly about the axis, and in fact the appearance of a tornado or a waterspout is that of a great writhing rope, the upward spiraling streams resembling its strands.

The axis of the tornado being an absolute vacuum, the only case known in nature, is cold. In direct contact with this delimiting surface is saturated air which is rapidly condensed. Hence, in direct contact with the axial vacuum is a shell of water or at least water drops with gaps of saturated air. This water is whirled upwards against gravity until it is thrown out at the top at some distance from the axis. The latent heat from the condensation supplies an additional ascensional force to the air about the core which causes it to whirl upwards in a steeper spiral than the core itself.

If we regard a tornado as having so little thickness that it may be regarded as a shell, there must be an equilibrium at any point between the general outer pressure of the atmosphere and the horizontal centrifugal force of the tornado. There are in reality two centrifugal forces, one due to the horizontal rotation, or $\frac{C^2}{r^3}\cos\varphi$, acting normally on the surface outwards and the centrifugal force due to the velocity upwards in a vertical plane through the axis, which acts normally on the surface inwards. This vertical centrifugal component, however, is small, especially in the lower part of the tornado, where the vertical radius of curvature is very large.

Neglecting this vertical component, therefore, we have practically the horizontal centrifugal component, $\frac{C^2}{r^3}\cos\varphi$, balanced by the general outside pressure of the atmosphere.

Now the atmospheric pressure is $p_0 e^{-\frac{z}{k}}$, where z is the height above the surface of the earth, p_0 the pressure at the surface of the earth, and k is the barometrical coefficient.

Or $\frac{C^2}{r^3}\cos\varphi = p_0 e^{-\frac{z}{k}}$. Since the angle φ is small, at least in the lower part of the tornado, the surface of the tornado in its lower part can be approximately represented by the formula $\frac{C^2}{r^3} = p_0 e^{-\frac{z}{k}}$, this equation representing a vertical section through the axis. Such a curve is shown in Fig. 22, and represents very nearly the form of a tornado. A tornado may occur either over land or water. When originating over the ocean, they are called waterspouts. Still, as we have before remarked, there seem to be certain differences in these phenomena, whether on land or water, according as they orignate in the middle or in the equatorial circulation.

In the polar circulation there are no cyclones or tornadoes, owing to a lack of aqueous vapor sufficient for their continued existence. Continuous gales are common, but these are not local phenomena, being shared in more or less by the whole circulation.

Occasionally the water in the core of a tornado after being carried up is discharged en masse, resulting in a so-called cloudburst which deluges the areas upon which it falls. It often happens that tornadoes in the middle circulation carry the water surrounding their axes to a height where the temperature is low enough to freeze it. Hence, hail storms are a frequent accompaniment of tornadoes. Hail is in general an indication of a whirl aloft, though when this whirl has not extended its foot to the ground, it is not called a tornado and its existence is otherwise unperceived and unknown. The axis of a whirl is not necessarily a straight vertical line, and in fact it frequently is not. When the upper part of the axis bends over so that particles of water at a distance from the axis are carried alternately from a region below zero to one above zero, it may happen that the hail stones show concentric layers of ice deposition, increasing their diameter with every whirl through the colder spaces.

A somewhat analogous phenomenon, but not a tornado, are the moving columns of dust which are sometimes encountered on deserts. These are probably started by two opposing currents aloft, as are tornadoes. An opening is thus formed in the upper colder layers through which the intensely heated air from the bottom can rise. The great difference of

temperature between the top and bottom may thus suffice to prolong their existence with a minimum amount of vapor condensation. But they wholly lack the violence of tornadoes, having little energy and not lasting long. Further, there is never a vacuum in the axis. They are thus analogous to, but not tornadoes. So, too, are thunderstorms, which consist of whirls in the upper layers, the axes often oblique, and thus giving rise to showers and hail.

A tornado rarely lasts more than two hours, during which time it may travel from 30 to 50 miles or more. The direction is invariably easterly in the middle circulation. The velocity near the axis may be greater than 500 miles an hour.

OTHER PHENOMENA OCCURRING IN THE ATMOSPHERE

SOUND

We have seen that any local change in the atmosphere creates a condition of unstable equilibrium which results in the passage of the disturbance from its origin to outlying points. We have hitherto dealt with disturbances on a grand scale. We are now to consider the results of sudden changes of pressure on a small scale.

Since a gas possesses inertia any sudden thrust against it will be resisted just as much as if it were a solid. The reaction will be equal to the action. Since it is elastic, the gas will be compressed between these two forces acting in opposite directions. Since the gas is free to move, it will move also in the direction of the force. The action of a thrust is, therefore, double. It sets the gas in motion and at the same time compresses it. Hence, we may divide the work done on the gas into two parts—the work done in compressing it and the work done in imparting to it its kinetic energy. This double energy imparted to the gas will at first be limited to the immediate vicinity where the disturbance arose, but as time goes on, the changed portions of the gas will shift their positions and the disturbance travels along as a wave of compression or rarefaction, or both.

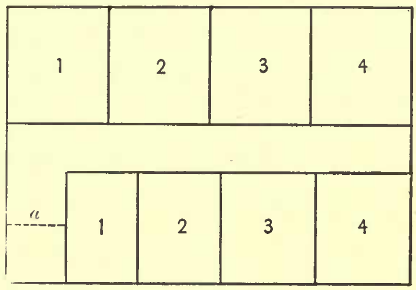


FIG. 23

Let us suppose the gas divided off into laminæ of equal thickness, 1, 2, 3, 4, etc., as represented in the upper part of Fig. 23. Let us further suppose that lamina 1 has been thrust by a force to the right and that the disturbance after a certain interval has extended to lamina 4, so that the bounding planes occupy the positions shown in the lower part of the figure. Let us suppose that at this instant the velocities of the laminæ are v_1, v_2, v_3, etc.

The pressure of each lamina on its neighbor to the right is partly static and partly kinetic. Thus the pressure of lamina 3 on lamina 4 is $p_0 + dp_3 + m\frac{dv_3}{dt}$, where dp_3 is the excess of the pressure of 3 over that of 4, and $\frac{dv_3}{dt}$ is the acceleration of 3 relatively to 4, p_0 being the general external pressure.

If we suppose that in an infinitesimal interval of time, dt, the laminæ have moved ds_1, ds_2, ds_3, etc., we have as the work of compression on lamina 4,

$$\left(p_0 + dp_3 + m\frac{dv_3}{dt}\right) ds_3.$$

Likewise the work of compression of lamina 2 on lamina 3 will be $\left(p_0 + dp_2 + dp_3 + m\frac{dv_2}{dt}\right) ds_2$, and the work of compression of lamina 1 on lamina 2 will be $\left(p_0 + dp_1 + dp_2 + dp_3 + m\frac{dv_1}{dt}\right) ds_1$.

Since we can neglect differentials of the second order in comparison with differentials of the first order, we can write for the work of compression on the several laminæ during the time dt

$$\left(p_0 + m\frac{dv_3}{dt}\right) ds_3 \quad \text{compressional work on 4.}$$

$$\left(p_0 + m\frac{dv_2}{dt}\right) ds_2 \quad \text{compressional work on 3.}$$

$$\left(p_0 + m\frac{dv_1}{dt}\right) ds_1 \quad \text{compressional work on 2.}$$

Now the sum of these infinitesimal works of compression is the total compressional energy of lamina 1, for we can suppose lamina 1 to be compressed by successive steps. Thus we can compress lamina 4 into lamina 3, then lamina 3 into lamina 2, and finally lamina 2 into lamina 1.

Hence, the total compressional energy of lamina 1 is the sum of the expressions we have written above. Or it is

$$p_0\,(ds_1 + ds_2 + ds_3) + m\sum_{0}^{v_1} v\,dv.$$

Now, $ds_1 + ds_2 + ds_3$ is the distance which lamina 1 has been compressed, or the amount of its compression. Call this distance c; then the total compressional energy of lamina 1 is

$$p_0\,c + \frac{m\,v_1^2}{2}.$$

Hence, at any point of a wave, no matter what its form or type, the compressional energy is equal to the normal pressure p_0 into the distance the lamina has been compressed plus its kinetic energy. In other words, its kinetic energy is equal to the work of compression done upon it *in excess* of the work of compression done by the general external pressure.

Since in a vibratory wave, during the passage of one complete wave, a lamina has moved forwards and back to the identical point and condition from which it started, it follows that the work done upon it by the general pressure is zero. Hence, for the complete wave the total kinetic energies are equal to the total compressional or potential energies.

A formal proof of this theorem is given in Lord Rayleigh's "Theory of Sound," par. 245, where it is stated:

"It follows that in a progressive wave of any type one-half of the energy is potential and one-half is kinetic," and again, what amounts to the same thing, "the total energy of the wave is equal to the energy derived from compressing its whole mass from its minimum to its maximum density, or to the energy of the whole mass moving with its maximum velocity." For a complete vibratory wave this is true. For a unidirectional wave, such as the compressional wave we have just studied, the statement does not apply. In such a case, as we have seen, the total compressional energy is equal to the total kinetic energy, plus the total work of compression of the general external pressure p_o. Or the total compressional energy of a unidirectional wave is equal to its total kinetic energy plus the normal pressure into the distance its extreme limiting plane (on the left) has moved during the time. In Fig. 23, it is easily seen that the sum of the compressions of all the laminæ, or Σc, is equal to the distance "a," which the extreme limiting plane on the left has been thrust by the force.

Let us suppose that a thrust has moved the adjacent gas a distance a and then ceases; also that during the time of the thrust the disturbance has extended to a distance l. After the wave has passed, each particle has been moved along a distance a and set down at rest in its new position.

Let us suppose that the average velocity of a particle during this time is v. Then while the disturbance has moved a distance l, the particle has moved a distance a. Call V the velocity of the wave. Then $V = \frac{l}{a} v$.

The gas of the wave occupies at normal pressure a volume l. Its total compressional energy is practically the same as that derived from compressing a volume l at normal pressure to a volume $l - a$. By thermodynamics, the work of performing such a compression adiabatically is

$$\frac{p_o l^k}{k - 1}\left[(l - a)^{1-k} - l^{1-k}\right].^*$$

a, the amplitude of the wave, is generally small, so that we can neglect higher powers. Expanding the above expression and neglecting powers of a above the square, we have

$$W = \frac{p_o l^k}{k - 1}\left[(l - a)^{1-k} - l^{1-k}\right] = p_o a + \frac{p_o k}{2 l} a^2.$$

* k is the ratio of the two specific heats.

The kinetic energy of the wave is $\frac{D\,l}{2}\,v^2 = \frac{D\,l}{2} \cdot \frac{V^2\,a^2}{l^2}$, where D is the absolute density, or mass per unit volume of the medium.

Now we know from our preceding theorem that the total compressional energy is equal to the total kinetic energy plus the expression $p_0\,a$.

Hence, $$\frac{p_0\,k}{2l}\,a^2 = \frac{D\,V^2\,a^2}{2\,l} \text{ or } V = \sqrt{\frac{p_0\,k}{D}}.$$

This is the velocity of propagation of the wave for amplitudes so small that higher powers than the square of the amplitude can be neglected. For a negative or expansional wave, that is, a wave arising from a thrust to the left, it is easily seen that the velocity is the same, provided we neglect higher powers of the amplitude than the square.

For a compressional or positive wave, if we consider the third power of the amplitude, we have

$$V = \sqrt{\frac{p_0}{D}\left(k + \frac{k\,(k+1)}{3} \cdot \frac{a}{l}\right)}.$$

Hence, the velocity of a compressional wave increases with the amplitude, and also with the pitch, or as a increases or l decreases, since the shorter the wave length l, the greater the number of pulses passing a given point in a given time.

For a negative or expansional wave,

$$V = \sqrt{\frac{p_0}{D}\left(k - \frac{k\,(k+1)}{3} \cdot \frac{a}{l}\right)}.$$

Hence, a negative wave travels more slowly than a positive wave, when the amplitude becomes appreciable, and the difference between their velocities increases with an increasing amplitude. This is further self-evident, since a positive wave is heated and a negative wave is cooled. It follows, therefore, that the velocity of a negative wave decreases with the amplitude and also with the pitch.

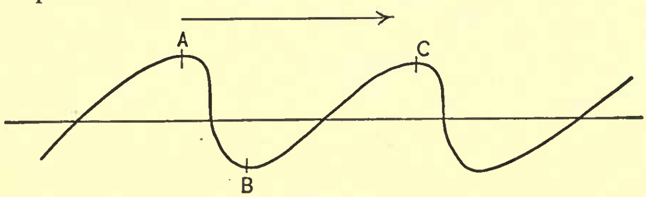


FIG. 24

These results have most important bearings. Since a vibratory wave is composed of a positive and a negative half, of equal amplitudes and following each other at equal intervals of time, it will be seen that the positive half will always tend to overtake the negative half in front of it, so that the wave will assume the form

and cannot be a sine wave, as is usually assumed. Sine waves, in fact, do not exist in nature. The crests tilt forwards, and the hollows tilt backwards as shown in the figure. This form of the wave has most important bearings on attractional results, as will be shown later.

As the amplitude increases, the difference between the velocities of the crests and hollows increases, until at length a point is reached where the crests break over into the hollows, precisely as water waves rolling up on a beach break. Mathematically expressed, discontinuities arise. The subject of discontinuities in wave motion has been treated at length by Riemann, Christoffel, Hugoniot, Hadamard, Lamb and others.

Practically, the result is that sustained rhythmical sounds of great intensity are impossible. When a tuning fork or a bell vibrates excessively, it loses the purity of its tone as well as rising in pitch. The waves break and become mixed up, losing more or less their rhythmical character. Ignorance of this principle has resulted in many attempts to construct powerful musical instruments for the purpose of producing very loud musical sounds which should carry to great distances. Organs blown by steam are an example. The result has invariably been a failure. The fabled music of the spheres must have been of rather small amplitude.

Where a vibratory wave has considerable amplitude and yet does not break, a compromise velocity is effected between the two halves, which is less than the standard velocity, $\sqrt{\frac{p_o k}{D}}$ of a small amplitude.

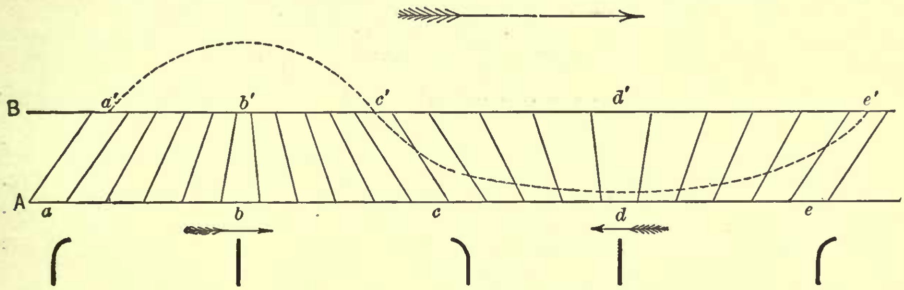


FIG. 25

To understand the intimate mechanism of a vibratory wave let us examine the accompanying Fig. 25. Let the line *A* represent a row of equidistant particles at rest. Then as a wave train sweeps through them, their positions will become changed and at a certain instant will be as shown in the line *B*. The wave is going in the direction of the large arrow. The

positions of a tuning fork as it swings are shown under the corresponding points of the wave. The section e will have been moved into the position e'. It is now farthest to the right of its position of equilibrium, i.e., its original position, and is momentarily at rest, and, therefore, of normal density. The sections following from right to left indicate what its state will be at successive instants. It is now swinging backwards, i.e., against the wave, and on arriving at d' is at its original position, but with a maximum expansion and, therefore, moving with a maximum velocity. At c' it is farthest to the left, having completed its excursion and come momentarily to rest. It now swings forwards, *with* the wave to the right, and on arriving at its original position at b' has a maximum density and velocity. At a' it has completed its excursion to the right and is at rest, with normal density, preparatory to executing its backward swing again. The wave curve indicates the pressures or profile of the wave.

The velocity of a compressional wave we have seen is

$$V=\sqrt{\frac{p_0 k}{D}\left(1+\frac{2\,(k+1)}{3\,!}\cdot\frac{a}{l}+\frac{2\,(k+1)\,(k+2)}{4\,!}\frac{a^2}{l^2}+\frac{2\,(k+1)\,(k+2)\,(k+3)}{5\,!}\frac{a^3}{l^3}\text{ etc.},\right.}$$

while that of an expansional wave is

$$V=\sqrt{\frac{p_0 k}{D}\left(1-\frac{2\,(k+1)}{3\,!}\cdot\frac{a}{l}+\frac{2\,(k+1)\,(k+2)}{4\,!}\frac{a^2}{l^2}-\frac{2\,(k+1)\,(k+2)\,(k+3)}{5\,!}\frac{a^3}{l^3},\text{ etc.}\right.}$$

The Laplacian value for the velocity, $\sqrt{\frac{p_0 k}{D}}$, is nearly 332.4 meters per second at 0° C. This is for very small amplitudes. We see that for large amplitudes the velocity may become very much greater than this. Thus, in firing cannon, if the experiment be suitably performed, the command "Fire" will be heard *after* the report. The heavy amplitude of the gun easily outstrips the weak energy of the voice.

Standing in the line of fire of a saluting six-pounder (smokeless powder), the author has, by some rough determinations, convinced himself that the velocity of the report was considerably greater than the Laplacian value.

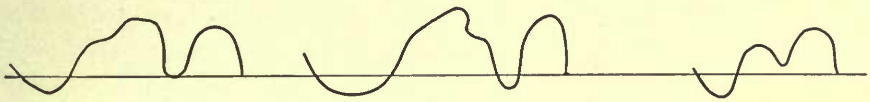
Vieille (R. Ac. Sc. 1898-99. Memorial des Poudres et Salpêtres, tome 10, pp. 177-260) has found in his experiments with explosive waves in long tubes of small caliber, velocities up to 1200 meters per second, and greater; which is about four times the Laplacian value, or the value usually given as the velocity of sound.

Within the bore of firearms—rifles and cannon—the compressional wave is transmitted from the breech to the muzzle with enormous velocities. On emerging from the muzzle into free space, the wave undergoes modifications which influence its velocity materially. The wave is launched out in a certain sense as a gaseous projectile. It immediately spreads out laterally and at the same time draws the air towards itself from the rear and sides. A wave of rarefaction, therefore, follows close behind the wave of compression. Further, the outgoing lateral wave of compression touches along a surface the incoming lateral wave of rarefaction, and along this surface circular vortices are formed. With black powder this smoke ring, exactly like the circular vortices blown by tobacco smokers, is usually observed attending the discharge of cannon.

We have seen that directly in front of the gun the velocity of the report is greater than the Laplacian value. Directly behind the gun it is less, since the sound heard here arises from a negative wave. The difference between the times when a report is heard from a saluting ship as the guns are fired towards and away from the observer, is very appreciable.

That an explosive wave of compression is followed by a negative wave, is rendered clear from the accompanying autographs of a small rapid-fire gun taken from MacKendrick's "Visible Speech." (Fig. 26.)

For a single explosion, the wave consists of a marked compressed or positive portion, followed by a negative wave of lesser amplitude.



MERGED EXPLOSIONS

FIG. 26

A positive wave of great amplitude, traveling in free space, has a tendency to lengthen and thus to distribute its energy over a greater space from front to rear, in addition to its inevitable geometric distribution which is inversely as the square of the distance traveled. The swifter moving more condensed parts crowd up towards the front, which thus becomes steep, and gives rise to an ever increasing distance between the front and rear. Thus gun-fire, which is heard as a short, sharp report near by, at a distance becomes a long, low rumble.*

We have hitherto supposed the compressions and rarefactions to be effected adiabatically. These may under certain circumstances take place

* An analogous phenomenon may be seen in a high waterfall. A mass of water may detach itself from the main body, when the lower part, which has a slight start over the upper part, continually increases the distance between them, until finally the mass breaks up into drops.

nearly isothermally. If c denotes the compression, the total work of compression, if it takes place isothermally, is $p_0\, l \int \frac{d\,c}{l-c}$, since $p = \frac{p_0\, l}{l-c}$.

Hence, $W = p_0\, l \log \left(\frac{l-c}{l}\right) = p_0\, l \log\left(1 - \frac{c}{l}\right)$. Developing the logarithm by Maclaurin's theorem and neglecting powers of c above the square, since c is small, $W = p_0\, l \left(\frac{c}{l} + \frac{c^2}{2\, l^2}\right) = p_0\, c + \frac{p_0\, c^2}{2\, l}$. The last term, we have seen, is equal to the total kinetic energy, or $\frac{p_0\, c^2}{2\, l} = \frac{l\, D\, v^2}{2}$ $= \frac{D\, V^2\, c^2}{2\, l}$, where v and V denote the same quantities as before. Whence $V = \sqrt{\frac{p_0}{D}}$. This is the value which Newton found for the velocity of sound on the assumption that the compressions and rarefactions were isothermal. Now, a single compressional wave must inevitably give up some of its heat of compression. Further, when the amplitude is very small and the wave of considerable length, the distinction between adiabatic and isothermal compressions tends to vanish. For short vibratory waves, no matter how small the amplitude, the compressions are, no doubt, practically adiabatic, but, as we have said, for long unidirectional waves of small amplitude, they may approach isothermal conditions. We have an example of this in an experiment which was performed upon a grand scale. About 10 A.M., local time, of August 27, 1883, the volcano of Krakatoa blew off its top in a culminating explosion. This started a compressional wave which circled round the globe no less than seven recorded times. The records appeared as notches in the tracings of the self-registering barometers of all the meteorological stations of the world.

For 29 stations the mean velocity from Krakatoa to them on the first lap was 707 English miles per hour. For 27 stations, from the first to the third passage, or after the wave had completely encircled the globe, the average velocity was 684 miles per hour. From the third to the fifth passage the average velocity was 682 miles per hour. From the fifth to the seventh passage the average velocity was 676 miles per hour. These numbers are taken from the "Report of the Royal Commission on the Eruption of Krakatoa."

The Newtonian velocity at 0° C. is about 640 miles an hour. If we suppose that the temperature of the tropical belt was 25° C., then a wave traversing this region would have a Newtonian velocity of 671 miles per hour, which is very nearly the last recorded velocity. The Laplacian value is 747 miles for 0° C., and 773 for 25° C. The velocities are thus very much less than the Laplacian value and approximate closely to the Newtonian value. There is every reason to believe that the succeeding oscilla-

tions, which were too small to be measured, approached the Newtonian value as a limit. But the Newtonian value is a positive limit. No disturbances in a gaseous medium can be propagated with a less velocity than $\sqrt{\frac{p_0}{D}}$, while, as we have seen, they may be propagated with very much greater velocities than the Laplacian value.

H. J. Rink (1873), in "Poggendorf's Annalen," Bd. 149, pp. 533-546, was the first to call attention to certain differences between explosive (unidirectional) waves and vibratory (bidirectional) waves. He pointed out that in the former there was a bodily transference of the gas, which remained in its new position all along the line, while in the latter each particle vibrated about a position of equilibrium, and, after the disturbance had passed, occupied precisely the same point it had originally.

Regnault partly recognized this, for he says: "On doit donc admettre qu'au moment du tir d'une arme à feu, le gaz comprimé qui s'en échappe, est lancé d'abord comme un projectile, qui imprime non seulement une compression, mais aussi un translation aux coudes d'air voisines. Ce dernier effet devient probablement insensible à une certaine distance, mais il doit troubler notablement la vitesse de propagation élastique dans le voisinage du départ. J'ai eu souvent occasion de reconnaître les effets de translation dans nos experiences surtout dans celles qu'ont été faites dans de tuyaux de petite section."

We have already called attention to the fact that the positive and negative halves of a vibratory wave have trouble in keeping step, or rather in keeping company. The natural velocity of the positive half is greater than that of the negative half, so that the former is continually crowding up on the latter. A compromise is at first effected by a distortion of the wave, the crests tilting forwards and the hollows tilting backwards. The profile of the wave is, therefore, like that represented in Fig. 25.

Now, if we consider a particle of matter imbedded in the medium, of a density exactly equal to that of the medium, it is evident that it will move to and fro with the vibrations exactly as if it were a part of the medium. There will be no resultant translation of the particle in one direction or another. The case is very different, however, if the particle be more dense or less dense than the medium. Let us consider the case of a denser particle. As the wave train sweeps over it, it will be actuated by two forces, viz., the pressure gradient of the wave and the streaming past it of the medium first in one direction and then the reverse. As the compressed part from B to A sweeps by, it will be urged to the right by an excessive gradient and also by the flow to the right of the condensed medium. From C to B, the medium, of less density than normal, will stream against it to the left, and it will further be urged to the left by a more gentle pressure gradient. But the important factor here is time. The compressed half will

urge it to the right for an appreciably shorter time than the expanded portion urges it to the left, and time triumphs in this case. A great force acting for a short time, as the blow of a hammer, will move a body less than a lesser force acting for a longer time, as the push of a man.

For a discussion of this problem, the reader is referred to the article "Gravitation" in *Popular Astronomy* for January, 1905. It can be shown that a denser body will be moved *against* the wave, while a less dense body will be carried along *with* the wave. The limiting velocity in the latter case is, of course, the velocity of the wave itself.

The following examples will serve to illustrate this peculiar wave action. If a balloon filled with carbonic acid gas be brought near a vibrating tuning fork, it will move towards the fork, i.e., against the wave train. If the balloon be filled with hydrogen gas, it will move away from the fork. And if the balloon be of the same density as the air, it will remain stationary. The author believes that the explanation of gravitation is closely connected with these phenomena.

Gravitation is not merely an attractive force between bodies; it may be attractive or repulsive according to circumstances. It is a force which cannot act through nothing. That is, it must produce its effects through the intervention of a medium, and we have every reason to believe that this medium is the universal ether. Radiant energy not only sets bodies into vibration, that is, it not only heats them, but it also exerts an attractive force on bodies denser than the ether. On the other hand, it exerts a repulsive force on bodies less dense than the ether, as in the case of comets' tails. The sun, by the waves which it sends through the ether, drives this tenuous material away with a velocity nearly equaling, or perhaps equaling, that of the waves themselves.

Further, the ether can exert no attractive or repulsive force on itself; that is, it cannot be mutually gravitative or repulsive. It is simply the bearer of waves. It, therefore, of itself has no weight or temperature, but it confers these attributes on other bodies. That it is a material substance it is impossible to doubt, since, as we have seen, it possesses both inertia and elasticity, the fundamental properties of all material substances. It is certainly compressible, else it could not transmit waves. It differs from ordinary matter chiefly in that its elasticity is very great, while its inertia is very small, but this difference is merely one of degree. That it is a uniform continuous body, in contradistinction to the segregation of ordinary matter, seems highly probable.

OPTICAL PHENOMENA

THE RAINBOW

Let Fig. 27 represent a magnified raindrop, and $A\ O$ the direction in which the sun's rays fall on it. Some of these rays will be refracted, suffer

a partial internal reflection and then be refracted out again. It is easily seen from the figure that the angle through which an incident ray will have been turned is $\pi + 2i - 4r$. Now as the parallel rays from the sun fall on different points of the circumference of the raindrop it is evident that the angle through which each is turned, or the total deviation from its original direction, will vary with the angle of incidence. We desire to determine the particular angle of incidence where this deviation is a minimum.

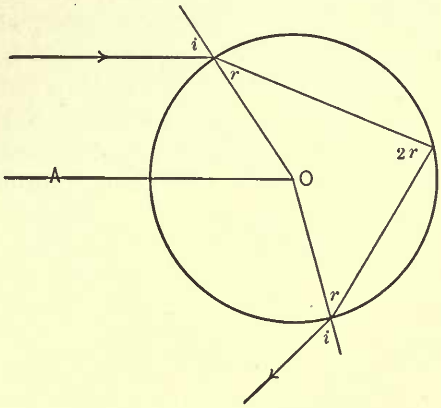


FIG. 27

Differentiating $\pi_1 + 2i - 4r$ with respect to i and equating to zero, we find that $\frac{dr}{di} = \frac{1}{2}$ is the condition. But by the law of refraction $\sin i = \mu \sin r$, where μ is the index of refraction. Hence, $\frac{dr}{di} = \frac{1}{\mu}\frac{\cos i}{\cos r}$

$$= \frac{\cos i}{\sqrt{1 - \frac{\sin^2 i}{\mu^2}}} = \frac{1}{2}.$$

The index of refraction μ, from air into water, is, for the middle part of the spectrum, $\frac{4}{3}$. Solving our equation, we find that $i = 59° 20'$ and that the minimum deviation is $138°$.

If now we stand with our back to the sun and the drop is moved about in a circle making an angle of 42° with a line drawn from the sun and prolonged through our head, it is evident that we shall see the ray of minimum deviation at all points of this circle. If we move the drop just outside this circle it is evident that no ray will now reach us, since the deviation of all rays will be greater than the angle between our line and the drop.

If we place the drop anywhere within the circle, we shall receive some rays from it in all positions. Even near the center of the circle rays will, theoretically at least, be reflected directly back to us. Now, in considering all the rays which fall upon the drop, it will be seen that a great many will be crowded into the position of minimum deviation, for as we increase the angle of incidence from zero, the deviation will change rapidly with the angle until we approach the angle requisite for a minimum deviation, when they will mark time; and on going beyond this angle they will retrace their steps. Hence, if we distribute drops all over the region of our circle, there will be a very faint illumination at all points within it; least at the center and gradually increasing as we approach the circle; increasing very much as we get

near the circle; and darkness at all points outside the circle. If now, while the sun is low, a shower is falling on the opposite side of the horizon, we shall have a moving picture. Each drop on entering our circle will for a moment flash out brightly and then become practically invisible. It is easily seen that for the red, the least refrangible rays, the circle will be largest, while for the blue, or most refrangible rays, the circle will be smallest. Hence, our circle will resolve itself into a spectrum with the red outermost and the blue innermost. The circle we have considered is called the first bow, and measurements show that its radius is 42°, as we have found theoretically. It is evident that this first bow can never be formed when the sun is higher than 42° above the horizon. We have hitherto considered only the rays striking on the upper half of the drop. Those falling on the lower half form a precisely similar but reversed primary bow, that is, with the vertex downwards, which might be seen by a man in a balloon far above the shower.

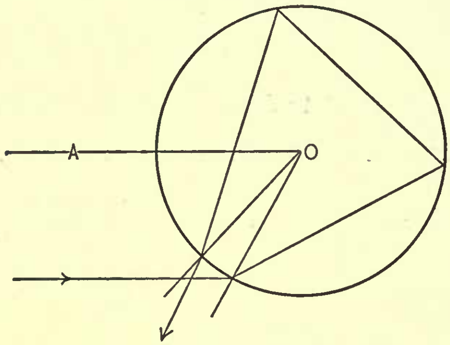


FIG. 28

We saw that the rays forming the first bow were twice refracted and once reflected, but on striking the inner surface, just previous to the last refraction, all the rays do not emerge. Some are reflected once more and then refracted, and theoretically this goes on indefinitely, but practically they become so weakened that it is not possible to detect any bows beyond the second, that is, one due to two refractions and two reflections.

By the same reasoning as previously applied, we find that this second bow is concentric with the first, but has a radius of 51°.

By the second reflection matters are reversed, so that now the red rays are on the inside and the blue rays on the outside. Further, the faint luminosity that we found within the first circle is now outside the second circle, while between the two circles there are absolutely no rays.

If the light forming the bows came from a point and was homogeneous, our circles would be thin rings, but as the sun has a very appreciable disc and the light is not homogeneous, the width of the bows is considerably increased. 42° 2′ and 40° 17′ would be the radii of the respective circles, for the first bow of red and violet rays proceeding from a point.

Very faint supernumerary bows are sometimes seen within the first bow. We saw that the faint rays illuminating the inner space were composed of rays which had suffered a greater deviation than the minimum. For every ray on one side of the minimum, there is a corresponding ray on the other

side which has the same deviation, and hence these rays interfere. Some of these interfering rays have a half wave length difference of phase, others a wave length, and so on, so that the colored rings which they form are regularly spaced. They are usually alternate green and purple. It is necessary for their production that the raindrops be extremely uniform in size. The distance between the supernumerary bows will be greater as the drops are smaller.

There are many disturbing factors to prevent the formation of a pure bow, such as diffraction, the reflection of light from clouds and from the drops themselves, etc., so that owing to overlapping we often have white bows.

We have seen that cirrus clouds are minute ice crystals floating in the air at very great heights. Ice crystals are either thin plates or upright prisms, but the predominant angle throughout all its forms is 60°. They may be hexagons, triangles or flower-shapes, but we always find the angles 60° and 90°. The refractive index of ice is 1.31, and the refracting angle is always 60° or 90°. We have seen that refracted rays are always crowded together in the direction of minimum deviation. Hence, the main effect of light refracted by ice crystals in the air will be seen in this direction, while the straggling rays will produce a much feebler illumination. By the same method we have used for rainbows, we find that light issuing from a celestial object and refracted by ice prisms of 60° should produce a circular halo about the object with a radius of 22°. Refracted by ice prisms of 90°, the radius of the halo should be 46°. For homogeneous light these haloes would have a width about equal to the sun's or moon's apparent diameter. Measurements of the ring or double ring sometimes seen about the sun or moon confirm the theory. It is well known that haloes have a radius of 22° or 46°, and they are spoken of as 22° haloes and 46° haloes. The minimum deviation is least for the red rays; hence, the inner border of the haloes will be red or at least reddish. For the same reason, the straggling rays (rays not of minimum deviation) will all be outside of the ring. The inner border, therefore, will be more sharply delimited and its color more perceptible than that of the outer border. The external portions of haloes are usually practically white.

We have supposed the rays to pass through the prisms at right angles to the edges. Much of the refraction, however, will be oblique, which contributes to destroy definition. When the crystals in falling tend to set in a particular direction, edgewise or lengthwise, and the sun is near the horizon, the horizontally refracted rays will be principally seen and a brilliant image of the sun may be perceived at the same level to the right or left, at an angle of 22°. These images are called Parhelia or Mock-suns. When the sun is some distance above the horizon, the refraction through the vertical prisms will be oblique, and hence the angle of minimum deviation greater than 22°.

Hence, as the sun rises higher the mock-suns gradually separate outwards, still, however, keeping the same apparent altitude as the sun. If the prisms lie mainly horizontally, the mock-suns will be directly above and below the sun, provided the axes of the prisms are perpendicular to the line joining the sun and the observer. As they usually lie in all horizontal directions, the result will be a series of mock-suns which form a new halo, touching the halo of 22° above and below where it is brightest, and gradually fading off on each side. Such haloes are called tangent arcs to the halo of 22°. Of course there may also be tangent arcs to the halo of 46°. We have supposed that the axes of the prisms are mainly directed in one direction in falling, the vertical or the horizontal, and this is probably the case for the brighter parhelia and tangent arcs, but it is not necessary. When the axes are distributed at random, the selective refraction which reaches the eye out of the large number of vertical and horizontal prisms may give rise to the phenomenon.

The subject is not ended here. Sometimes the ice prisms are terminated with hexagonal pyramids, and these may produce haloes. The mock-suns themselves may produce secondary haloes, so that the results may be very complicated. Coronæ are diffraction phenomena and are produced when the sun or moon is seen through a mist or a cloud. They are concentric colored rings surrounding these bodies and are equally spaced. The width of each ring depends upon the size of the drops of water, which must be very uniform. The smaller the drops, the wider the rings. Being diffraction rings, the red is on the outside, instead of on the inside, as in the haloes.

REFRACTION DUE TO NON-HOMOGENEITY OF THE AIR

The refractive index of air at 0° C. and 760 mm. pressure is 1.000294 or about $1 + \frac{1}{3400}$. Careful measurements show that, for ordinary ranges, the excess of this value over unity is proportional to the density. Or the refractive index of air is $1 + \frac{p}{760} \cdot \frac{273}{273 + t} \cdot \frac{1}{3400}$ (1), where the pressure is expressed in mms. of mercury and the temperature is centigrade. Considering a small portion of the earth's surface a plane and the density of the air equal everywhere at the same level, but changing with the level, it is evident that a ray of light will travel in a straight line from one point to another at the same level. A ray directed obliquely upwards, as it passes through layers of varying density, will suffer refraction, and as the densities change continuously, not abruptly, the path of the ray will be a curved line. It may happen, therefore, that one ray will proceed in a direct line from one point to another at the same level, and that another ray will reach it by a curved line, or there may be several possible curved paths. The absolute

index of refraction of a substance is the ratio of the velocity of light in a vacuum to its velocity in the substance.

Since we can resolve an infinitesimal portion of the path of a ray into a horizontal component, which does not change its velocity, and a vertical component which does, it is evident that the acceleration (or retardation) of the ray is in a vertical direction. Denoting the vertical direction by n, we have

$$\frac{d\,v_n}{d\,t} = \frac{d\,v_n}{d\,n} \cdot \frac{d\,n}{d\,t} = v_n \frac{d\,v_n}{d\,n} = \frac{d\left(\frac{v_n^2 + v_h^2}{2}\right)}{d\,n},$$

since v_h, the horizontal velocity, does not change. Hence, the acceleration is measured by the rate of change of the kinetic energy per unit length in the direction of the vertical. We say kinetic energy because we suppose a small particle to represent the motion of the ray. It is evident that the path must be symmetrical with respect to the vertex, which is half way between two points at the same level, from the principle of reversibility.

Since the curvature must be very slight, the radius of curvature has practically a vertical direction for every portion of the path, so that by ordinary mechanical principles we can equate the centrifugal force to the vertical acceleration. Now, the velocity at any point is proportional to μ, the index of refraction.

Hence, $\dfrac{\mu^2}{\rho} = \dfrac{d\left(\frac{\mu^2}{2}\right)}{d\,n}$, where ρ is the radius of curvature, or $\dfrac{1}{\rho} = \dfrac{1}{\mu} \cdot \dfrac{d\,\mu}{d\,n}$.

If we suppose the temperature near the earth to be constant, say 0° C., then the density will decrease slightly for small changes of level.

By (1)
$$\frac{d\,\mu}{d\,n} = \frac{dp}{dn} \cdot \frac{1}{3400\,p_0}.$$

$$\therefore \rho = \frac{\mu}{\frac{d\,\mu}{d\,n}} = \frac{1 + \frac{p}{3400\,p_0}}{\frac{dp}{dn}\,\frac{1}{3400\,p_0}} = \frac{3400\,p_0 + p}{\frac{dp}{dn}}. \qquad (2)$$

Under normal conditions the pressure near the surface of the earth decreases $\dfrac{1}{7997.8}$ for each meter, since the height of a homogeneous atmosphere of the surface density would be 7997.8 metres. Thus, $\dfrac{dp}{dn} = -\dfrac{p_0}{7997.8}$. Substituting this value of $\dfrac{dp}{dn}$ in (2), we have $\rho = -3401 \times 7997.8$. (3)

The minus sign shows that the path curves downward. The curvature of the earth, which is unity divided by its radius, is about four times this amount. If, then, we introduce the curvature of the earth, we see that it will not be possible for a ray to proceed in a straight line between two points on the same level, but one point will be seen from the other by means of the curved ray we have just investigated. For this reason all points surrounding the observer, whether on the horizon or above it, seem to be raised above their proper position. The position of the point directly overhead is, of course, not changed. In ascending in a balloon likewise, the point directly underneath is judged correctly, but the rim of the horizon is very much raised and the impression conveyed is that of a huge bowl. Balloonists generally remark that the earth appears to be hollowed out beneath them.

Since $\mu = 1 + \frac{p}{p_0} \cdot \frac{273}{273 + t} \cdot \frac{1}{3400}$, if the temperature changes with the level, we have $\frac{d\mu}{dn} = \frac{1}{3400}\left(\frac{273}{(273+t)\,p_0} \cdot \frac{dp}{dn} - \frac{p}{p_0}\,\frac{273}{(273+t)^2} \cdot \frac{dt}{dn}\right)$. Whence $\frac{1}{\rho} = \frac{1}{3400}\left(\frac{1}{p_0} \cdot \frac{dp}{dn} - \frac{1}{273} \cdot \frac{dt}{dn}\right)$ very nearly. Or we can write very nearly, $\frac{1}{\rho} = -\frac{1}{3400}\left(\frac{1}{7997} + \frac{1}{273}\,\frac{dt}{dn}\right)$.

If the temperature decreases as we ascend, then $\frac{dt}{dn}$ is negative and further if $\frac{dt}{dn} = -\frac{273}{7997}$, then there will be no curvature of the ray, which is equivalent to saying that the density of the air remains constant. This, under normal conditions at the surface, would correspond to a fall of 1° C. for about every 30 meters. The real average rate near the surface is something like 1° for every 200 meters. If, numerically, $\frac{dt}{dn} > \frac{273}{7997}$, then the curvature of the ray will be upward and we shall have the phenomenon of the mirage, which may be studied by turning Fig. 29 upside down. Objects are here seen depressed below the horizon and give the observer the impression that he sees them by reflection, although the image is not inverted as it would be by reflexion. However, it is impossible to tell whether clouds are inverted or not, and even for objects which might be tested for inversion, the mind does not stop to do this, but judges at once that a sheet of water lies before it. Travelers on a desert where the air is intensely heated close to the surface are frequently deceived into believing that water lies before them.

The pressure, no matter how the temperature varies, usually changes at a practically constant rate. If the temperature also changes at a constant rate with the height, the path of the ray will be an arc of a circle, since ρ will be constant. Given, therefore, an object *A B*, Fig. 29, and knowing the

rate of change of temperature with the height, we can determine ρ. Then through the points O and A we draw a circle of radius ρ. This will determine the path of a ray from A to O. Likewise, the same circle passing through O and B will determine the path of a ray from B to O. Hence, an observer at O will see the object raised and erect. This is the case of ordinary atmospheric refraction. Since the rate of decrease of temperature

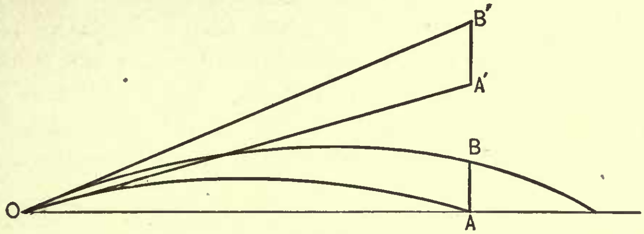


FIG. 29

with height is generally much less than the limiting value $\frac{273}{7997}$, i.e., since the air in general becomes less dense with the height, all objects are raised above their proper positions. In navigation and astronomy this necessitates corrections for refraction. While these corrections may be applied with some degree of accuracy when the body has a considerable altitude, it is easily seen that when the body is near the horizon, they cannot be applied with any degree of trustworthiness.

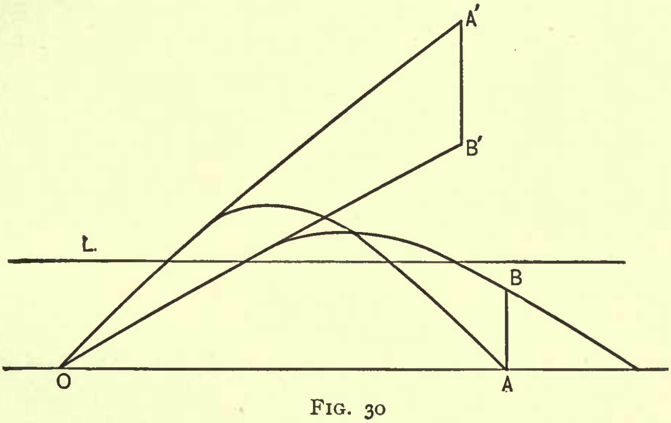


FIG. 30

If the lower air has a constant density up to a certain level L, but beyond this a constantly decreasing density, it will be seen that the rays will pursue paths as indicated in Fig. 30. In the lower layer the paths will be straight, while above they will be curved as before. The path of the ray from the lower point A will be steeper and its vertex higher than that of the ray from B. Hence, their paths must cross and the image will be inverted. Generally inverted images are magnified, while erect ones are diminished

in size. Under certain conditions, as in Fig. 30, a ship may be seen directly through the uniform lower layers and at the same time an inverted image of it, high in the air and usually magnified. With a combination of layers, some of which have a decreasing density, others an increasing density, while others still have a uniform density, the different possible results are greatly increased. Thus a ship may be seen directly and over it a series of images, one above the other, some inverted and some erect, in every possible sequence.

Objects below the horizon, which under normal conditions could not be seen by ordinary refraction, are sometimes raised temporarily by these abnormal refractions, so that they may appear high in the air, erect or inverted, as the case may be. Sailors call this "looming." As all these effects are due to refraction, the images have colored borders, though the colors are not particularly noticeable.

ELECTRICAL PHENOMENA

Since we do not know what electricity is, it will be impossible to explain these phenomena in the complete manner in which other phenomena have been explained. We must content ourselves with a description of them and the various surmises which have been advanced.

The atmosphere always contains free electricity, which in the vast majority of cases is positive, and increases with the height above the ground. The difference of potential may be as much as 600 volts per meter (Exner). The potential appears to vary inversely as the vapor pressure and consequently increases with cold. Thus certain measurements gave a potential of 325 for 2.3 mm. of vapor pressure, 116 for 6.8 mm., and 68 for 12.5 mm. of vapor pressure. In fact, the electrification of the atmosphere seems to depend chiefly upon the water constituents, vapor and ice. Generally the greatest amount of electricity is observed when the barometer is highest. Clouds are reservoirs of electricity which are positive, unless by inductive influence from a more strongly charged cloud they become partly negative. With a clear sky the electrification of the upper regions is always positive: if negative it is due to the influence of clouds.

The potential increases with condensation, for if 1000 vapor particles each possessing the same charge coalesce to form a single droplet, the diameter of the droplet will be only ten times that of one of the original particles, and since the capacity is equal to the radius, its capacity will be only ten times as great. But the charge will be one thousand times that of a single particle; hence, the potential will be one hundred times as much.

The electricity of the ground is generally negative with regard to the atmosphere, but a cloud induces electricity of the opposite kind on the earth's surface directly below it. Hence, a cause of earth currents. From high exposed points electricity is always passing between the earth and the atmosphere. From mountain peaks this discharge is sometimes luminous. Lem-

ström observed in winter a flame-like appearance from the tops of two mountains, 800 and 1100 meters high. Occasionally also, clouds which are discharging electricity to the earth become luminous.

The cause of the separation of the two electricities, speaking merely in the conventional sense, seems to be due to friction between the earth and air and between portions of the air itself which are under different conditions. Faraday showed that the friction between minute particles of water and dry air is an abundant source of electricity. Armstrong's hydro-electric machine demonstrates the same thing. The middle and polar circulations which rub against each other in opposite directions at their border should be a source of electricity.

When the potential between two clouds or a cloud and the earth becomes great enough, the insulation will be broken down and lightning results. Flashes of lightning are often a mile in length and sometimes as much as five miles in length. They are usually zigzag, as after proceeding a certain distance in one direction, the resistance increases rapidly, and they turn in another direction, though probably not as sharply as it appears. The electromotive force necessary to produce a spark a mile long in air at normal pressure has been estimated as something over 3,000,000 volts. However, the droplets are intervening conductors, and in reality the lightning does not take a single leap. Hence, it is possible that discharges often take place where there is no very great difference of potential. The quantity of electricity, however, is always very great.

There are several varieties of lightning. The ordinary zigzag flash is almost always between earth and cloud and is similar to the discharge of a Leyden jar. Sheet lightning is more of the nature of a brush discharge between two clouds. Ball lightning is a peculiar form which is not yet understood. It consists of a globe of fire, from an inch up to eighteen inches in diameter, which falls to the earth so slowly that it can be followed by the eye. These balls often rebound on striking the earth, or they may explode with a loud report. Planté, with an enormous battery (great quantity and tension), has imitated them. He used as electrodes two sheets of blotting paper moistened with distilled water, which is a very bad conductor. Placing these a short distance apart, he was able to produce a small globe of fire which rolled about between them. His explanation was that they were globes of intensely heated, rarefied and dissociated gases, which preserved their form by means of an envelope, in a quasi-spheroidal state, which was practically non-heat-conducting. The electric tension between earth and cloud thus produces a discharge through the rarefied globe precisely as we do in the laboratory through a vacuum tube. The fact that flashes with a beaded appearance, resembling in fact a rosary, have been seen, seems to bear out this explanation.

St. Elmo's Fire—brush discharges from the ends of masts and spars seen

in bad weather are similar to the discharges we have already spoken of from mountain tops.

Thunder is due to the heating and tearing effect of the electricity as it passes. Near by it is short and sharp; far off it is a long rumble, due to the lengthening and flattening of the sound wave with distance. Of course, to this is added the effect of echo from the clouds and the ground. Thunder is not heard at very great distances, the energy of a thunder clap not being as great as that of heavy gun fire. The cannonading at Waterloo was heard in England. Thunder has never been heard at such a distance, fifteen miles being the usual limit.

The aurora is in all probability a brush discharge between the earth and aqueous particles (ice) in the higher layers. The height at which this phenomenon takes place has been greatly overestimated. Balloons have occasionally been in the midst of the light as well as observers at the surface of the earth. The following is taken from the International Cyclopædia: "If we consider the aurora as a discharge through aqueous vapor or other gas, then we have nothing to do with the gaseous character as such, but with the aqueous component only, and at moderate altitudes the density of the aqueous vapor is so slight that it must act as a very light gas, similar to that present in a vacuum chamber. As regards the height of the aurora above the earth's surface several methods have been devised for calculating it: but all trigonometrical calculations based upon most careful observations seem to show that the definite features that we see in the aurora are perspective phenomena and that the calculation of their height cannot be safely made by the method of simultaneous observations at two stations. In fact, the argument for the existence of the aurora quite close to the earth's surface is too strong to be ignored."

It is quite possible that they do not extend much above seven miles, the region of aqueous vapor.

The belt of greatest frequency of auroras has been found to be a curve passing through Point Barrow, the northern portion of Hudson's Bay, the southern tip of Greenland, Iceland, the North Cape, and the arctic coast of Russia and Siberia. For the southern aurora there is also such a circle of maximum frequency. That is, observers to the north of this line see more auroras to the south than to the north, while observers to the south see more to the north. This line corresponds somewhat with the border between the middle and polar circulations, where, we have seen, there is reason to believe that an excessive amount of electricity is generated. In the polar regions, the fall of potential with altitude is thirteen times greater in summer and eighteen times greater in winter than at the equator. Hence, an electrical phenomenon, which depends upon the magnitude of this fall of potential, must be more intense in winter and high latitudes than in summer and in the torrid zones. The discharge may be considered as passing from the earth to the

aqueous vapor above, and as not becoming luminous (visible to the eye) until some definite altitude (rarefaction) is reached. The rays (currents) have a direction in general identical with the magnetic lines of force springing from the earth's surface. This is a position of equilibrium for a movable current. It is well known that a current capable of moving, which cuts a line of magnetic force, will be moved in a direction at right angles to both. If the ray is parallel to the line of force, it will not be moved, except perhaps twisted spirally about the line. When a ray moves at an angle with the line of force, it will be displaced and the quivering usually noticed in the streamers is probably due to such a motion imparted by the magnetic field of the earth.

An observer looking along an outgoing ray, i.e., end on, will not see anything. Consequently in the direction of the lines of magnetic force there will appear overhead a less illuminated spot in the heavens which is called the corona of the aurora. By perspective the rays from the outlying points will appear to converge towards this point.

If there are ice crystals in the corona, the rays of light reflected nearly directly backwards will at some points interfere, at others reinforce themselves, and a flower-shaped or rayed figure will be produced, consisting of alternating dark and bright bands issuing from the center. The edges of these bands will be colored. The general color of the aurora, as its name indicates, is reddish, and is practically the same as that produced by a brush discharge through rarefied nitrogen. The characteristic yellow-green line in its spectrum is due to krypton, one of the heaviest gases in the atmosphere.

The rays may be visible at the surface of the ground, but usually there is a definite interval between their lower extremities and the ground. This level is probably dependent upon the degree of rarefaction necessary for luminosity, just as in a Geissler tube a certain degree of rarefaction is necessary before luminous effects are produced. Thus, it is probable that the greater the intensity of the discharge, the nearer will the streamers appear to be to the earth, and, as we have already seen, at times they are seen springing from the earth.

The structure of the atmosphere, that is, the exact arrangement of its different layers and the amount of its vaporous constituent at different points, must determine at any instant the precise configuration of the aurora. The upper air is far from homogeneous and is rather to be characterized as streaky.

"A remarkable characteristic, moreover, met with, is that when the direction of such wind changes the change may be perfectly abrupt. It has, indeed, been recorded by scientific balloonists that they find in regions where winds of different directions pass that one appears actually to drag against the surface of the other, as though tolerating no interval of calm or transition;

and yet a more striking fact is that a very hurricane may brood over a placid atmosphere with a clear-cut surface of demarcation between calm and storm."—*J. M. Bacon.*

Each layer is moving over the other, so that the conductivity in a given direction is continually changing, and the appearance of an aurora may be likened to an image which is projected through a large number of photographic plates which are continually moving relatively to each other. At certain points in the lower layers the rays are completely cut off, while at other points gaps allow them to pass through, just as gaps in a cloud let beams of the sun through. The aurora is thus a projection or picture of the electrical conductivity of the atmosphere. It has frequently been remarked that the aurora has the appearance of draperies blowing in a wind. This is probably an actual fact, since we know that aloft there are always winds blowing with hurricane velocity. We might liken the atmosphere to a gauze curtain, which, as it waves in the wind, is lit up at different points in rapid succession.

It has been noticed that the aurora moves slowly as a whole, generally from east to west, but sometimes from west to east. It seems probable that this is due to the general motions of the polar and middle circulations. When the aurora is on the polar side of the border between these two circulations, its motion should be towards the west. It would seem, therefore, that the aurora is formed more frequently in the polar circulation.

De la Rive has attempted to explain this phenomenon by an ingenious experiment known as "De la Rive's Experiment." It is, in fact, an application of the principle that a movable current in a magnetic field is thrust aside in a direction perpendicular to the current and to the lines of magnetic force. We have seen from the foregoing that this principle may explain the quivering of the rays, but it cannot explain the east-west motion, since the rays are nearly coincident with the lines of force.

An aurora is usually of enormous extent, as is shown by the magnetic perturbations and earth currents observed coincidently with it at widely different points, even where it may not be visible. The luminosity simply shows that where this occurs the vibrations are large enough to affect the eye. Doubtless if the eye were sensitive enough it could perceive light at all points where the streams exist.

Lastly a connection undoubtedly exists between auroras and sun-spots. The frequency of auroras increases with sun-spots, there being an eleven-year period of wax and wane for both. It would seem as if magnetic lines of force were shot out from the sun along the axes of the spot vortices, and these lines being cut by a moving conductor—the earth—produce currents perpendicular to the direction of motion, i.e., in a north-south direction. This would, of course, lead to brush discharges at the poles.

The auroral discharge frequently produces a sound like the rustling of wind, exactly such as may be heard in the brush discharge of an electrical

machine. This shows that the observer is actually in the midst of the discharge, as otherwise it could not be heard.

MECHANICAL FLIGHT

The problem of mechanical flight naturally resolves itself into two possible methods—that of a mechanism which has the same or about the same density as the air, and therefore floats in it, and that of a mechanism which is heavier than air and is kept afloat by its own energy, which it expends against the inertia of the air in a downward direction. In other words, it drives the air downwards, thereby overcoming its inertia, and the reaction from overcoming this inertia of the air urges it upwards. Since this reaction is equal to the mass of air displaced into its acceleration, or the rate at which its downward velocity is increased, it follows that by applying sudden and powerful thrusts against the air, a heavy body may be supported or even raised.

Of the first form of airship, it need only be said that in still air a balloon, provided that its volume be kept constant, floats at some definite level, which is where its density is equal to that of the surrounding air. By expending energy in thrusts against the inertia of the air in a horizontal direction, it will move the air in that direction, and will move itself in an opposite direction, and the total mass of the air moved into the distance it has moved will be equal to the mass of the balloon into the distance it has moved. For such a dirigible balloon, of course, it will be necessary to have an elongated or cigar shape in order to reduce to a minimum the resistance of the air in front of it. Generally, unless it is moving with the wind and with the exact velocity of the wind, it will have to overcome a virtual wind, which is the geometric difference between the velocity of the actual wind and the velocity of the balloon. In order to overcome the pressure of this virtual wind it will have to possess some rigidity, as otherwise it would be crushed. We have seen that the wind pressure on a plane surface opposed to it normally was proportional to the area of the surface into the square of the velocity of the wind. The formula, $P = .003\ V.^2\ S$, expresses the pressure in pounds per square foot when the velocity is given in miles per hour. The pressure increases rapidly with the velocity. Hence, it would be impossible for a dirigible to go against a very strong wind, even provided it had unlimited power. We have seen that a hurricane blowing eighty to one hundred miles an hour often destroys permanent buildings of the most solid construction. No dirigible, therefore, could withstand such exceptional pressures. It could, however, go *with* a gale, without any great trouble, only a certain amount of airmanship being necessary to overcome the vertical components.

Dynamically, the aeroplane is more interesting. If a plane be moved

horizontally through the air with a slight angle of elevation, it will continually thrust down a certain mass of air, which by its reaction will tend to prevent the plane from falling. To support the plane without rising or falling, the energy expended will have to be precisely that necessary to thrust down a mass of air, such that its reaction will be equal to the weight of the plane. Thus to keep an aeroplane afloat requires the constant expenditure of energy, and a considerable amount at that.

It has been supposed that birds can soar without the expenditure of any energy, and that it might be possible to construct an aeroplane which, once started, would maintain itself in the air without the further expenditure of any energy. The idea is, of course, erroneous, as all birds are continually falling. A leaf upheld by a vertical current is really falling through this air and at a considerable velocity. The resistance to its fall through this air is equal to its weight, and hence relatively to the earth it appears stationary. So, when we see a bird remaining at the same level, practically without moving a feather, we may be sure that it is falling relatively to the air around it, or, in other words, it is supported by a vertical current of which it is taking momentary advantage.

So, too, an aeroplane is continually falling, and it falls through a distance equal to the vertical distance between its front and rear edges, in the time it takes for the rear edge to move up to where the front edge was. It is the resistance to this fall which supports it. When the engine of an aeroplane is stopped, it glides on to be sure, but the energy now expended in displacing the air downwards is drawn from its kinetic energy, and as the velocity decreases, this is not sufficient to keep it aloft and it sinks towards the earth.

We have seen that the thrust between the aeroplane and the air is equal to the mass of air displaced into its acceleration at any instant. The air is not only pushed down by the under surface, but it is also pulled down by the upper surface. At the front edge, where the air is dead, this change of velocity, starting from zero, is most sudden, and therefore the reaction greatest. At the rear edge, since the air is now moving rapidly downwards, the acceleration is least. If the velocity of the air here is exactly equal to the rate at which the surface is moving downwards (along its incline), there will be no reaction upwards. If the velocity of the air downwards is greater than this, then there will be a negative reaction, or this air will pull the rear edge downwards. Hence, beyond a certain point it is a disadvantage to make the plane too wide, as, outside of the increased weight, we may convert a thrust upwards into a thrust downwards. We see then that the thrust against the under surface is not the same at every point, but is greatest at the front edge and least at the rear edge, so that the center of pressure does not correspond to the center of surface or center of gravity of the plane, being in advance of the latter. The result is that a couple is always at work tending to bring

the front edge up and the whole plane at right angles to the direction of motion.

In a lateral direction, however, we gain a distinct advantage in increasing the edge of the plane, since the lifting thrust is directly proportional to the length of this edge. Since the front edge engages only dead air, the limit of the length of this edge can only be set by consideration of practical construction, rigidity, etc. Large birds have great length of wing from tip to tip, but very little width. If it were not for this principle, we may be sure they would have been made with short, broad wings. An advantage is gained in curving the supporting surface so that the rear edge shoots the air directly downwards. The front edge then grips dead air, and this grip will be more or less maintained by the curve. Birds' wings are all curved.

Thus we see that it is necessary to construct aeroplanes of great length laterally, but small width fore and aft. However, as too great an extension laterally is inconvenient, if not impracticable, the most advantageous construction is probably a series of steps sloping backwards, so that we obtain a great length of front edge for cutting dead air. By this arrangement each plane engages practically undisturbed air, which would not be the case if the planes were vertically over each other.

We have seen that the maximum thrust comes upon the front edge where the air is dead, and that a couple always exists tending to turn the plane flatwise to the wind. A card always turns in falling, so as to bring its surface perpendicular to the line of fall. It balances for an instant in this position, when it slides off again in another direction, but shortly brings up by presenting its surface again, and thus by a series of swaying movements, between which it comes nearly to rest, gradually falls to the ground. The center of pressure moves quickly and seemingly erratically over the plane, but actually depends upon the velocity and angle of inclination. It would, for this reason, be clearly impossible to construct an aeroplane with a single plane, since, however we might ballast it fore and aft, it would be in unstable equilibrium. A slight increase or decrease in the velocity would tilt it up or down.

A great deal of trouble has been taken to calculate mathematically the center of pressure for a given velocity, angle of elevation, shape of plane, etc., but all this has little practical bearing. Very slight changes in any of these quantities will cause a considerable and in practice unforseeable shifting of the center of pressure. The problem of fore and aft equilibrium can only be solved by the use of two planes, well separated in a fore and aft direction. It does not matter then at what point the pressure comes in each plane. There is a forward pressure and a rear pressure, and by regulating one of these by a lever, equilibrium can always be maintained. Birds effect this equilibrium by means of their head, and legs and tail, chiefly the latter.

Lateral equilibrium is effected in much the same way. By raising the

tip of one plane and depressing the other, increased pressure is brought to bear on one side and decreased pressure on the other. This couple, acting through a long arm, turns the aeroplane laterally, so that equilibrium can always be effected by the aeronaut. Birds secure lateral equilibrium in the same manner. By the turn of the tip of a wing they right themselves. Since they have the power of moving the long wing feathers separately, this is easily accomplished.

It seems difficult to set limits to the possible development of the aeroplane in the future. By increasing the number of planes and setting them at different levels, preferably sloping backwards, so that in case of accident they might break the fall by opposing a maximum surface to a vertical descent, a single airship is practically multiplied into several as regards lifting power. This would mean that many passengers might be carried instead of one. The power, however, would have to be multiplied in like proportion, and this brings us to the present immediate difficulty—the motor.

Gasoline engines are now made weighing only a few pounds per horsepower, but it seems unlikely that this will be the motor of the future. As the power increases, the weight of a gas engine, at best, increases in a rather greater proportion; they are far from reliable and are lacking in flexibility, especially as the size increases. It would seem preferable that the energy should be *stored* in a separate reservoir instead of being producd in the cylinders. Steel flasks are now made of no great weight capable of withstanding a pressure of two thousand pounds to the square inch. If the explosions were produced in such a reservoir, the necessity for compressing the gas previous to explosion with the accompanying loss of stroke and energy would be avoided. The necessity of cooling the cylinders would also be done away with and the weight of the water used in cooling would be saved. This stored energy could be used in a turbine or reciprocating engine. Reciprocating engines have been constructed weighing less than a pound per horsepower. With improvements in metallurgy there is no reason to doubt but that some alloy may be found for the blades of turbines capable of withstanding the action of very hot gases. The improvement of motors, therefore, lies in the concentration of very great energy in a very small weight.

What we have said about planes applies to propellers, for propellers are planes. They must be long and narrow and a slight curve is advantageous so as to increase the grip of the posterior half. Since the velocity is proportional to the distance from the center, the inner surface may be dispensed with as not producing an effect commensurate with its weight.

The helicopter, where the planes of a propeller are used as the supporting planes, is impracticable in its ordinary form. By this method an inverted cyclone is produced in the air. Hence, the air after a few turns is moving down *with* the surface which seeks to gripe it. There is no dead

air on which to obtain a grip, except at the very beginning. Hence, the somewhat paradoxical result has been found experimentally that the lifting power does not increase with the velocity of the propeller and the energy expended, but may even decrease.

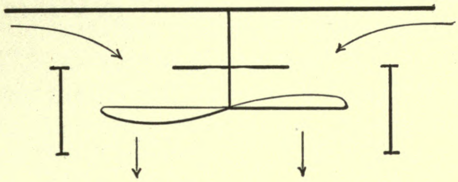


FIG. 31

By shielding the upper surface of the screw, as in Fig. 31, the air entering from the sides only in a horizontal direction may be suddenly shot down and thus becomes practically dead air, for the upward thrust is equal to the mass of air thrust *down* into the acceleration of this thrust.

The general principle holds that the upward thrust on the aeroplane is proportional to the square of the difference between the velocities of the plane and the air. Hence, an aeroplane cannot sail with the wind and with the same velocity as the wind. It is a matter of relativity. When an aeroplane sails with and against currents of varying velocity, the thrust given by the propellers will have to be varied; all of which shows the advantage of great reserve power and flexibility.

APPENDIX

THE GYROSCOPE

In Fig. 32 let us suppose that $G I$ is a disc capable of spinning about the axis $O G$, which passes through its center G. Further the axis $W O G$ is capable of turning about the fixed point O in any direction. We can abolish gravity by fixing a counterbalancing weight W at the other end of the axis, so that the system will move exactly as if gravity did not exist. We shall first give the disc a spin in a counterclockwise direction, so that the point I on the disc will be coming up through the page. Now, let us hit the system a smart tap, so that it will tend to revolve about the point O directly downwards through the page. That is, it will start to revolve about the axis $O P$, which is perpendicular to $O G$. It is now left to itself, no further forces acting on it, and the problem is to determine the subsequent motion.

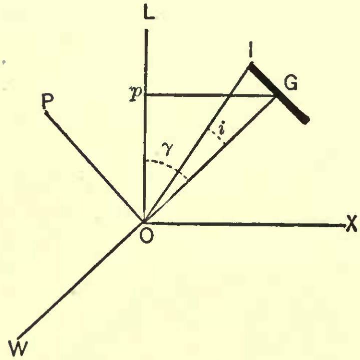


FIG. 32

We have here a double rotation, and according to our definition, it is this which constitutes a gyroscope. The points on the disc to the left of G are coming up through the page, and the whole system is moving bodily downwards through the page. Now, there must be some point to the left of G where these two opposite velocities exactly counterbalance each other. Let us suppose that the point I on the disc is this particular point. For the first instant of motion, therefore, the system will move as if it had only a simple rotation about the axis $O I$. We shall call this axis the Instantaneous Axis. We shall call the constant angular velocity with which the disc is spinning ω, and we shall suppose that the blow imparted to the system an angular velocity μ about the axis $O P$.

Since the point I remains for the first instant in the page and does move during this instant, we have

$$G I \omega = O G \mu. \quad (1)$$

We do not say that the system will continue to rotate about the axis $O I$, for such is not the case. We only say that at the first instant it will start to do so. Now, the moment of momentum of the disc about the axis $O G$ is

$M k^2 \omega$, where M represents its mass and k is its radius of gyration. This simply means that the *quantity* of motion in the disc would be the same if all its mass were concentrated into a thin ring at the distance k from the center, and this ring rotated with the same angular velocity ω. The moment of this quantity about the axis $O\ G$ is found by multiplying it by k.

Now, it is evident that the moment of momentum about an axis is a quantity which can be resolved into a component about any other axis by multiplying it by the cosine of the angle between the two axes. Let us draw through the point O two lines $O\ X$ and $O\ L$ mutually perpendicular. Drop a perpendicular from G to p on $O\ L$, and call the angles $G\ O\ I$ and $G\ O\ L$, i and γ respectively. Let the angular velocity of G about $O\ L$ be Ω and about $O\ I$, ω_i.

Now, the point G is passing through the page with a velocity $O\ G \sin i\ \omega_i$, and the disc is at the same time rotating about this point G, its center of gravity, with angular velocity ω.

Then its moment of momentum about the line $O\ X$ is $M \cdot \overline{O\ G}^2 \sin i. \cos \gamma\ \omega_i - M k^2\ \omega \sin \gamma$. (2)

The first term represents the moment of momentum of the mass, supposed to be concentrated at its center of gravity, about the axis $O\ X$, and the second term is the component about this axis due to the spin. They are given opposite signs because they rotate in opposite directions about $O\ X$.

It is clear that the moment of momentum about $O\ X$ will vary as we change the direction of this line. But we can always find one direction, and one only, in the plane $G\ O\ I$, such that this quantity will vanish. We have only to choose the angle γ so that the two partial moments we have just found shall be equal. In other words, the condition is that

$$M \cdot \overline{O\ G}^2 \cdot \cos \gamma \cdot \sin i \cdot \omega_i = M \cdot k^2 \cdot \sin \gamma \cdot \omega. \quad (3)$$

We see then that there will be absolutely no turning movement about such a line $O\ X$. In other words, the point G will start off in a direction parallel to the plane drawn through $O\ X$ and perpendicular to $O\ L$. But once started in this direction, since no further forces act upon it, it must forever afterwards move parallel to this plane.

Since the point G must remain at a constant height above this plane, it follows that the axis $O\ G$ will maintain a constant angle with the line $O\ L$; in fact, describe a cone about it. This line $O\ L$ we shall call the Invariable Line. It is evident that the instantaneous axis $O\ I$ will always remain in the plane $G\ O\ L$, so that the subsequent motion of our gyroscope is completely determined. It would be the same as if we imagined a cone $G\ O\ I$ fixed to the gyroscope rolling upon a cone $G\ O\ L$ fixed in space. This is the motion of a gyroscope started off with an initial impulse and then left to move without the action of any further forces. The axis $O\ G$ will continue revolving about

the invariable line $O\,L$ at a constant angle γ and with a constant angular velocity Ω. The system, as a whole, will at any instant be turning about an instantaneous axis $O\,I$ in the plane $G\,O\,L$, and this axis $O\,I$ will make a constant angle i with the axis $O\,G$.

Let us denote the moment of inertia of the disc about the axis $O\,P$, or $M \cdot \overline{O\,G}^2$ by A, and its moment of inertia about the axis $O\,G$ by C. From (3) we have, then, $A \sin i \cos \gamma\ \omega_i = C\ \omega \sin \gamma$. Now, $\omega = \omega_i \cos i$; hence, $A \tan i = C \tan \gamma$. Knowing, therefore, the mass of the gyroscope and the initial velocities of the spin and turn, the whole subsequent motion of the system is determined.

We can, instead of representing the motion as the rolling of one cone upon another, equally well represent it by supposing a surface of revolution to be attached to the gyroscope and allowing this surface to roll upon a plane perpendicular to $O\,L$ and tangent to the surface. The point of contact between the surface and the plane will trace two circles, one on the surface, the other on the plane. We shall, however, have to select our surface so that the angular velocity of the point of contact in the circle on the plane will be Ω, while its angular velocity in the circle on the surface will be ω. Poinsot, who first investigated this motion, called the circle on the surface the Polhode, and the circle on the plane the Herpolhode.*

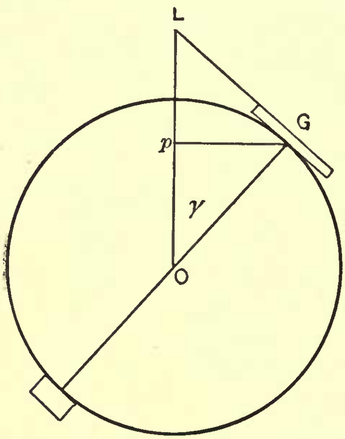


FIG. 33

Let us now draw a line $G\,L$, Fig. 33, perpendicular to $O\,G$. This line will evidently describe a cone tangent to a sphere drawn about O wtih the radius $O\,G$. The path of G will be a circle on this sphere. Since the gyroscope is revolving at a constant rate and distance from the point L, it follows that the centrifugal and centripetal forces directed away from and to this point must be equal. The centrifugal force tending to throw G away from L is

$M \cdot \overline{O\,G} \cdot \sin \gamma \cos \gamma\ \Omega^2$. Hence, in the motion we have considered, a force must be developed acting in the direction $G\,L$ and equal to $M \cdot \overline{OG} \cdot \sin \gamma \cdot \cos \gamma\ \Omega^2$. This force we shall call the gyroscopic force.

Since $A \tan i = C \tan \gamma$ and $\Omega \sin \gamma = \omega \tan i$, we may write

$$M \cdot \overline{O\,G} \cdot \sin \gamma \cos \gamma\ \Omega^2 = \frac{M}{\overline{O\,G}}\ k^2 \sin \gamma\ \Omega\, \omega.$$

But $\Omega \sin \gamma$ is the angular velocity with which the center G is turning in its path about the fixed point O. In other words, at any instant

* Poinsot. Théorie nouvelle de la rotation des corps, 1834 and 1852.

the gyroscopic force is always directed at right angles to the plane of motion of the gyroscope, towards the pole. Further, this gyroscopic force is equal to the mass into the square of the radius of gyration, into the angular velocity of the spin, into the angular velocity with which it is turning in its path about its fixed point, divided by the distance of its center of gravity from the fixed point. This is the fundamental property of a gyroscope from which all others can be easily derived. We shall now be able to understand all the peculiarities of motion of such a body under all circumstances.

To recapitulate, then, we have found *this* to be the fundamental property of a gyroscope, viz., that if we turn a spinning disc about a fixed point in its axis in any direction (or plane), there will immediately be set up a force pulling upon it in a direction at right angles to the direction in which it is moving. If the disc were not spinning, its path would be in the direction of the force moving it, but the spinning sets up a force at right angles to its path at any instant. We have seen that if we start it off in any direction and then allow no further outside forces to act upon it, it will continue to circle about a fixed line $O\ L$, and that the natural deflecting force is continually pulling it away from the great circle it otherwise would describe on the sphere. A small spin would only deflect it slightly, so that it would describe a circle nearly as large as the great circle it otherwise would describe, and tangent to this great circle at the point of starting.

From the previous discussion, we see that with a very great spin, i and γ both become very small, and therefore the result of striking a heavy blow on a rapidly spinning gyroscope would be to cause it to oscillate rapidly around a very small circle passing through its initial position. Here we find to a certain extent a confirmation of the popular idea that a gyroscope tends to preserve its plane of rotation. *But a gyroscope has no tendency to preserve its plane of rotation.* It changes its plane of rotation readily, only in a manner peculiar to itself. It is rather startling to find that with an infinitely great spin, we might strike our gyroscope a heavy blow with a hammer and yet be unable to move it. But this is merely a limiting case, and the proper way to regard it is that a gyroscope offers no resistance to a change of plane, but changes it in the manner we have just investigated and now understand.

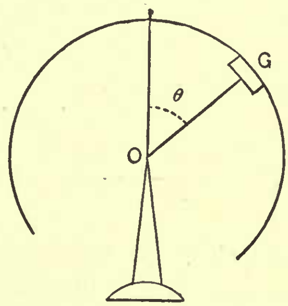


FIG. 34

We come now to consider the result of a constant and constantly directed force, such as gravity, upon a spinning body.

Let ϑ, Fig. 34, be the angle which the axis $O\ G$ makes at any instant with the opposite direction to gravity, and ψ the angle which a vertical plane through O and G makes with the initial vertical plane. The angles ϑ and ψ express the

colatitude and longitude of the point G at any instant. We shall represent the distance $\overline{O\,G}$ by l, and suppose the disc spinning in a counterclockwise direction. Taking the south and east directions as positive, we can write from the fundamental principle of the gyroscope.

$$l \sin \vartheta \, \ddot{\psi} = \frac{k^2\,\omega}{l}\,\dot{\vartheta} \;(1) \text{ and } g \sin \vartheta - l\,\ddot{\vartheta} = \frac{k^2\,\omega}{l}\,\dot{\psi} \sin \vartheta. \qquad (2)$$

Integrating (1) we have $\dot{\psi} = \dfrac{k^2\,\omega}{l^2} \log \dfrac{\tan \frac{\vartheta}{2}}{\tan \frac{\vartheta_o}{2}}$, (3)

where ϑ_o is the initial colatitude.

Substituting (3) in (2), we have

$$g \sin \vartheta - l\,\ddot{\vartheta} = \frac{k^4\,\omega^2}{l^3} \log\left(\frac{\tan \frac{\vartheta}{2}}{\tan \frac{\vartheta_o}{2}}\right). \sin \vartheta.$$

Multiplying by $\dot{\vartheta}$ and integrating,

$$-g \cos \vartheta - \frac{l\,\dot{\vartheta}^2}{2} = \frac{k^4\,\omega^2}{l^3}\left[\log \sin \vartheta - \cos \vartheta \log \frac{\tan \frac{\vartheta}{2}}{\tan \frac{\vartheta_o}{2}}\right] + K.$$

If the gyroscope starts from rest, $K = -g \cos \vartheta_o - \dfrac{k^4\,\omega^2}{l^3} \log \sin \vartheta_o$.

Hence, $\dfrac{l^2\,\dot{\vartheta}^2}{2} = \dfrac{v_p{}^2}{2} = l\,g\,(\cos \vartheta_o - \cos \vartheta)$

$$+\left(\frac{k^2\,\omega}{l}\right)^2\left[\log \frac{\sin \vartheta_o}{\sin \vartheta} + \cos \vartheta \log \frac{\tan \frac{\vartheta}{2}}{\tan \frac{\vartheta_o}{2}}\right], \qquad (4)$$

where v_p denotes the polar velocity.

$l\,g\,(\cos \vartheta_o - \cos \vartheta)$ represents the work done on the gyroscope by gravity, while $\dfrac{v_p{}^2}{2}$ is the kinetic energy of the polar velocity, and $\left(\dfrac{k^2\,\omega}{l}\right)^2$ $\left[\log \dfrac{\sin \vartheta}{\sin \vartheta_o} - \cos \vartheta \log \dfrac{\tan \frac{\vartheta}{2}}{\tan \frac{\vartheta_o}{2}}\right]$ is the kinetic energy of the horizontal velocity. Besides the initial value ϑ_o one other value of ϑ will cause v_p to vanish. The vertical movement of the gyroscope is, therefore, confined between these limiting values.

The gyroscope then at first begins to fall vertically, but immediately a horizontal motion is superadded. By the principle of energy, the total velocity at any instant will be proportional to the square root of the vertical drop,

but this is distributed between the vertical and horizontal components in respectively decreasing and increasing proportions, until finally the motion is all horizontal. After this it begins to rise, until at the original level it comes to rest again, and the process is repeated over and over again.

We shall next investigate the path which the point G traces. In order to simplify matters we shall consider only a very small portion of the sphere which may be taken as practically a plane. Hence, $l\vartheta$ and $l\chi$ will be rectangular coordinates within its limits, where $\chi = \psi \sin \alpha$. In this small area we can consider the moment of gravity constant and equal to $l\, g \sin \alpha$. Our equations of motion now assume the following simplified forms:

$$l\,\ddot{\psi} = \frac{k^2\,\omega\,\dot{\vartheta}}{l} \quad (5) \quad \text{and } g \sin \alpha - l\,\ddot{\vartheta} = \frac{k^2\,\omega}{l}\,\dot{\psi}. \quad (6)$$

Integrating, we have $l\,\dot{\psi} = \frac{k^2\,\omega}{l}\,\vartheta$ (7) and $g \sin \alpha\, t - l\,\dot{\vartheta} = \frac{k^2\,\omega}{l}\,\psi$ (8), since we assume the origin of coordinates to be where the body starts from rest.

If we write the equations

$$l\,\psi = \frac{g \sin \alpha}{K^2}\,(K\,t - \sin K\,t), \qquad (9)$$

$$l\,\vartheta = \frac{g \sin \alpha}{K^2}\,(1 - \cos K\,t), \qquad (10)$$

where $K = \frac{k^2\,\omega}{l^2}$,

we can easily see, by differentiating and substituting the values for $\sin K\,t$ and $\cos K\,t$, that (9) and (10) are the integrals of equations (7) and (8). Now equations (9) and (10) represent a cycloid with its base horizontal, convex downwards, and having a cusp at the origin of coordinates. Of course the point G cannot leave the sphere and a cycloid is a plane curve, but in the small part of a sphere which we can consider to be plane, it traces a part of a cycloid. Every portion of the path is, therefore, a portion of some cycloid in a plane tangent to that particular point of the sphere. It follows that the point G in tracing its path upon the sphere moves in the quickest time possible under the circumstances from one point of its path to another. For the cycloid is the curve of quickest rise and fall, and from any one point to another near it the gyroscope moves in the shortest time possible.

The equations of a cycloid are usually written $x = a\,(\varphi - \sin \varphi)$ and $y = a\,(1 - \cos \varphi)$, where a is the radius of the generating circle and φ is the angle which a radius makes at any time with its initial position. The radius of our generating circle, and therefore the cycloid, is quite large if ω be small, but if ω be large the generating circle becomes very minute. With a rapid spin, therefore, it would be possible to trace a large number of cycloids even in the small area to which we have restricted ourselves.

The radius of the generating circle is $g \frac{\sin \alpha}{K^2}$, or $g \frac{\sin \alpha \, l^4}{k^4 \, \omega^2}$. The time of falling through one of these cycloids is found by making Kt equal to 2π, and therefore is $\frac{2\pi}{K}$ or $\frac{2\pi \, l^2}{k^2 \, \omega}$.

We see that the time of one of these vibrations is independent of the actuating force or g.

We now have a complete knowledge of the motion of our gyroscope. If the spin be small, there is a considerable rise and fall which is easily observed. It here traces out a path which we might call a spherical cycloid, because every element of it is part of a definite cycloid, which could be drawn upon the plane tangent to the sphere at that element. With a very rapid spin, the rise and fall cannot be observed by the eye, though the ear can take up the humming caused by the minute vibrations. The motion here is the same as if the extremity of the axis were attached to the circumference of a very minute wheel and this wheel were rolled along on the under side of a parallel of latitude. The axis, therefore, carves out in space a cycloidally fluted cone with the cusps upward.

If at the instant of starting we could give the gyroscope an initial horizontal velocity, such that its upward deflecting force was exactly equal to the component of gravity, then there would be no rise and fall, and the axis would describe a smooth cone. This would be equally the case with a small or a great spin. If the spinning ceases, the cycloid degenerates into a great vertical circle and the motion becomes simply that of a pendulum passing at every swing through the point vertically under the point of support. But it will never pass through this point as long as it possesses the slightest spin.

We now see clearly why a gyroscope, or top, supports itself in seeming defiance of gravitation. As soon as it acquires a vertical angular velocity downward, normal deflecting forces are set up which pull it away from this direction, and the direction of its path being continually changed, it is eventually pulled up to the level from which it started. This is repeated in rapid oscillations over and over again. If we constrained the gyroscope to move in a vertical plane, as by placing the extremity of the axis in a vertical groove, it would swing like a pendulum, exactly as if it were not spinning, provided we do not take into account the friction caused by the horizontal pressure.

We have seen that the fundamental property of a gyroscope is that $\frac{M k^2 \omega}{O G} \dot{\vartheta} = M f$, where f represents an acceleration perpendicular to the plane of $\dot{\vartheta}$. This means that the normal turning moment about the point O is $M k^2 \omega \dot{\vartheta}$.

Now, if we turn a gyroscope about its own center of gravity, $O G$ will become zero and f mathematically infinite. But the product $f . O G$ still

remains finite, and is the turning moment of a couple exerted in a plane perpendicular to the plane of $\dot{\vartheta}$, or the plane in which we turn it, and passing through the axis. This moment is numerically equal to $M k^2 \omega \dot{\vartheta}$.

We have seen that a gyroscope progresses a distance equal to the circumference of the generating circle of its cycloid, or $\frac{2 \pi g \sin \alpha}{K^2}$, in the time $\frac{2 \pi}{K}$. It will, therefore, require a time equal to $\frac{2 \pi k^2 \omega}{g l}$ for a complete revolution, which is independent of the inclination of its axis.

SOME MEMORABLE HURRICANES

The first experience of Europeans with a West Indian hurricane was in the latter part of 1495. Columbus was at that time at the little settlement of Isabella, on the north coast of San Domingo, making preparations to return to Spain. "When the ships were ready to depart a terrible storm swept the island; it was one of those awful whirlwinds which occasionally rage within the tropics, and which were called 'uricans' by the Indians, a name which they still retain. Three of the ships at anchor in the harbor were sunk by it, with all who were on board; others were dashed against each other, and driven mere wrecks upon the shore. The Indians were overwhelmed with astonishment and dismay, for never in their memory or in the traditions of their ancestors had they known so tremendous a storm."*

The Great Storm of November 25, 1703 devastated nearly the whole of England. "Fifteen sail of the line, with Admiral Bowater and all his crews, with several hundred merchantmen were lost. London appeared like a city which had sustained a protracted siege, whole streets being destroyed, and several thousand individuals buried beneath the ruins. Eight thousand seamen perished." An annual sermon is still preached in London on November 25th, in commemoration.

The hurricane of August 31, 1772 destroyed ten thousand lives at Martinique.

The hurricane of October 10, 1780 called "The Great Hurricane," was of great extent and extraordinary violence, as is shown by its rapid northing. It proceeded from Trinidad, by way of Santa Lucia, to Bermuda. A large number of British and French men-of-war foundered. The French governor of Martinique made the laconic report, "The ships disappeared."

The hurricane of April 1, 1782 destroyed all the prizes taken by Rodney, together with an immense number of merchantmen and nearly all the men-of-war convoying the fleet.

The hurricane of September 16, 1782 is known as Admiral Graves' disaster. "The cyclone continued at N. W., and before it left the hapless fleet, the whole of the men-of-war, except the *Canada,* including the flagship, and all the merchantmen, had foundered. So large was the proportion of mer-

* Washington Irving.

chantmen that this is supposed to be the greatest naval disaster on record. Upwards of three thousand seamen alone are computed to have perished."

The Atlantic coast storm of March 11, 1888, which came up from the Bahamas, and which is known as "The Great Blizzard," destroyed nearly all the coast shipping at sea.

The great Samoan hurricane of March 16, 1889 wrecked and stranded six men-of-war and eight merchantmen, with a large loss of life.

The Porto Rican hurricane of August 8, 1899 destroyed three thousand lives. (Fig. 35.)

The Galveston hurricane of September 8, 1900 destroyed the city of Galveston, and with it six thousand lives.

The inundation wave accompanying the tremendous cyclone of June, 1822 destroyed, at Burisal and Backergunge at the mouths of the Burrampooter and Ganges, upwards of fifty thousand souls.

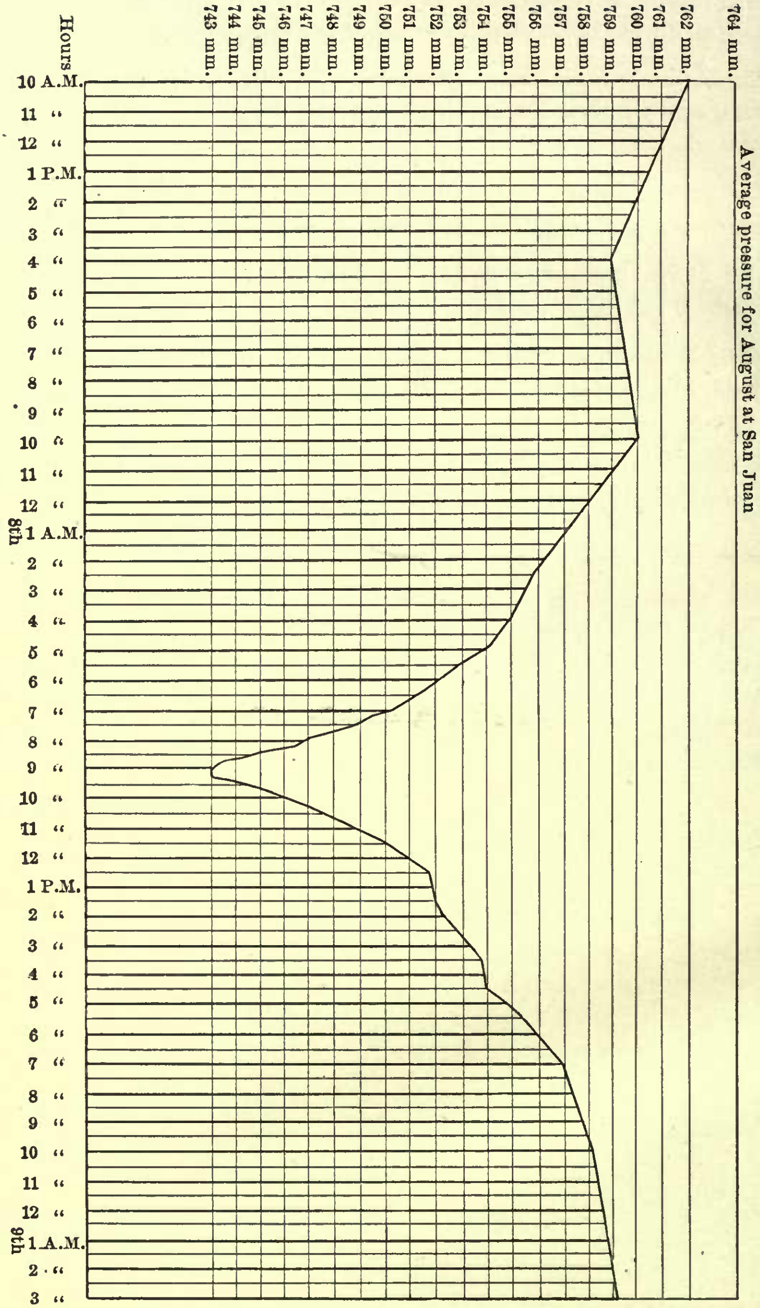


FIG. 35.—Barograph of Porto Rican Hurricane of August 9, 1899, taken at San Juan, by MR. J. A. CANALS, Civil Engineer.

VELOCITY OF THE EARTH AT VARIOUS LATITUDES

Miles Per Hour

Lat.	Naut.	Stat.	Lat.	Naut.	Stat.
0°	900	1037.5	31	773	890
1			32	764.7	780.6
2			33	756.3	871
3			34	747.7	861
4			35	738.7	850.8
5	897.6	1033 6	36	729.7	840.3
6			37	720.4	829.6
7			38	710.8	818.5
8			39	701.1	807.3
9			40	691	795.9
10	887.4	1022	41	680.8	784.2
11	884.5	1018.6	42	670.5	772.2
12	881.4	1015	43	660	760
13	878	1011	44	649.2	747.6
14	874.3	1007	45	638 1	734.8
15	870.4	1001.4	46	627	721.9
16	866.2	997.6	47	615.6	708.9
17	861.7	992.4	48	603.9	695.5
18	857.1	987	49	592.2	681.9
19	852.1	981.3	50	580.2	668.2
20	847	975.3	51	568.2	654.3
21	841.5	969	52	556	640
22	835.8	962.4	53	543.3	625
23	829.8	955.5	54	530.7	611.1
24	823.5	948.3	55	517.9	596.4
25	817	940.8	56	504.9	581.4
26	810.3	933	57	491.8	566.4
27	803.2	925	58	478.5	551.1
28	796	916.8	59	465.1	535.6
29	788.5	908	60	451.6	520
30	781	899.2			